Optimization and linear programming

Frames 1 to 83

Learning outcomes

When you have completed this Programme you will be able to:

- Describe an optimization problem in terms of the objective function and a set of constraints
- Algebraically manipulate and graphically describe inequalities
- Solve a linear programming problem in two real variables
- Use the simplex method to describe a linear programming problem in two real variables as a problem in two real variables with two slack variables
- Set up the simplex tableau and compute the simplex
- Use the simplex method to solve a linear programming problem in three real variables with three slack variables
- Introduce artificial variables into the solution method as and when the need arises
- Solve minimisation problems using the simplex method
- Construct the algebraic form of the objective function and the constraints for a problem stated in words

Optimization

1

An *optimization problem* is one requiring the determination of the *optimal (maximum or minimum) value* of a given function, called the *objective function*, subject to a set of stated restrictions, or *constraints*, placed on the variables concerned.

In practice, for example, we may need to maximise an objective function representing units of output in a manufacturing situation, subject to constraints reflecting the availability of labour, machine time, stocks of raw materials, transport conditions, etc.

Linear programming (or linear optimization)

Linear programming is a method of solving an optimization problem when the objective function is a *linear function* and the constraints are *linear equations* or *linear inequalities*.

In this Programme, we shall restrict our considerations to problems of this type that form an important introduction to the much wider study of operational research.

Let us consider a simple example, so move on to the next frame

2

A simple linear programming problem may look like this:

$$\begin{array}{lll} \text{Maximise} & P = x + 2y & (\textit{objective function}) \\ \text{subject to} & \left.\begin{array}{l} y \le 3 \\ x + y \le 5 \\ x - 2y \le 2 \\ x \ge 0;\ y \ge 0 \end{array}\right\} & (\textit{constraints}) \end{array}$$

The last two constraints, i.e. $x \ge 0$ and $y \ge 0$, apply to all linear programming (LP) problems and indicate that the problem variables, x and y, are restricted to non-negative values: they may have zero or positive values, but NOT negative values. These two constraints are often combined and written $x,\ y \ge 0$ – or omitted altogether since they are taken for granted in all LP problems.

Before we proceed, we will take a brief look at linear inequalities in general.

On, then, to Frame 3

3

Linear inequalities

In most respects, *linear inequalities* can be manipulated in the same manner as can equations.

(a) Both sides may be increased or decreased by a common term, e.g.

$$2x \leq y+4 \quad \therefore\; 2x-y \leq 4$$

(b) Both sides may be multiplied or divided by a positive factor, e.g.

$$4x+6y \geq 12 \quad \therefore\; 2x+3y \geq 6$$

But NOTE this:

(c) If both sides are multiplied or divided by a negative factor, e.g. (-1), then the inequality sign must be reversed, i.e. $\geq$ becomes $\leq$ and vice versa.

Here, then, is a short exercise.

Exercise

Simplify the following inequalities so that each right-hand side consists of a positive constant term only.

(a) $3x-5 \leq 4y$
(b) $2(x+2y) \leq -8$
(c) $4x-6y \leq -10$
(d) $2x+3 \geq -(y+4)$
(e) $-(x-3y+5) \geq 2x+4y-6$

Check the results in the next frame

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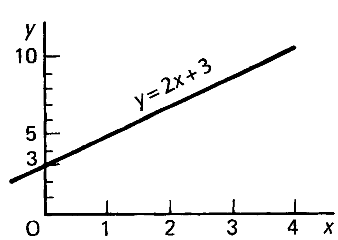
(a) $3x-4y \leq 5$
(b) $-x-2y \geq 4$
(c) $-2x+3y \geq 5$
(d) $-2x-y \leq 7$
(e) $3x+y \leq 1$

Graphical representation of linear inequalities

Consider the inequality $y-2x \leq 3$. We can add $2x$ to each side, so that $y \leq 2x+3$.



The equation $y=2x+3$ can be represented by a straight line dividing the x–y plane into two parts.

For all points on the line,

$$y=2x+3.$$

For all points below the line,

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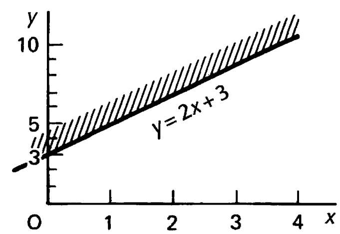
$$\boxed{y < 2x + 3}$$

$\therefore\ y \leq 2x + 3$ indicates all points on or below the straight line, but excludes all points above it. We can indicate this exclusion zone by shading the upper side of the line.



Arguing in much the same way, $x - 2y \leq 2$ can be rewritten as $y \geq \dfrac{x}{2} - 1$ and we can draw the line $y = \dfrac{x}{2} - 1$ and shade in the exclusion zone

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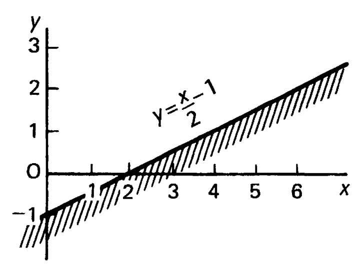
$$\boxed{\text{below the line}}$$

i.e.



Example 1

The problem we quoted earlier in Frame 2 was

$$\begin{array}{lll} \text{Maximise} & P = x + 2y & (\textit{objective function}) \\ \text{subject to} & \left.\begin{array}{l} y \leq 3 \\ x + y \leq 5 \\ x - 2y \leq 2 \\ x \geq 0;\ y \geq 0 \end{array}\right\} & (\textit{constraints}) \end{array}$$

Now, on a common pair of x and y axes, we can represent the five constraints and shade in the exclusion zone for each. We then have the composite diagram

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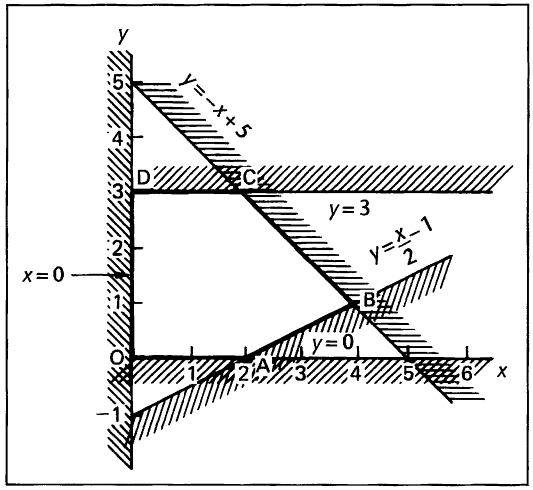

The coordinates of all points on the boundary of the polygon OABCD, or within the figure so formed, satisfy the system of constraints. The set of variables for each such point is called a *feasible point* or *feasible solution* and the figure OABCD is the *feasible domain* or *feasible polygon.*

Note these definitions.

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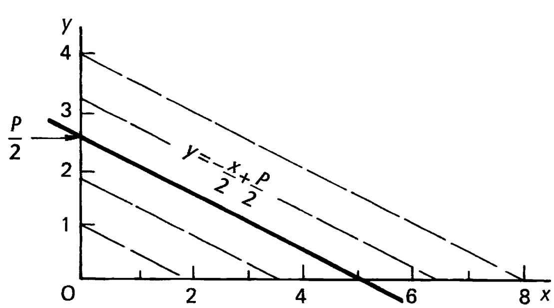
Our problem now is to find the particular point within this domain that makes the objective function $P = x + 2y$ a maximum. The equation can be rewritten as $y = -\frac{x}{2} + \frac{P}{2}$ and this represents a set of parallel lines with different values of the intercept $\frac{P}{2}$.



If we draw one sample line of this set to cross the feasible polygon we have just obtained, we get, using $P = 3$

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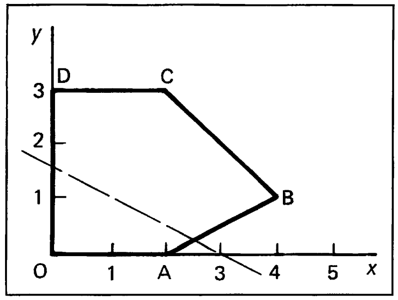
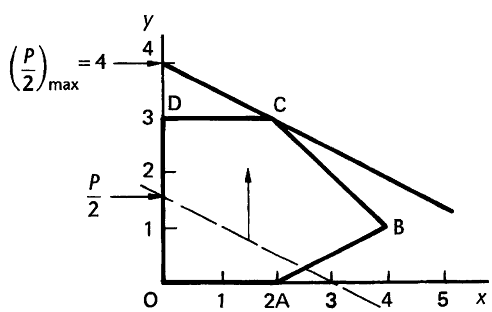


We can increase the value of P and hence raise the objective line up the page until it passes through the extreme point C. Any further increase in the value of P would take the line outside the feasible polygon and hence fail to conform to the given set of constraints.



In this example, then, point C gives the optimal solution.

From the graph it can be seen that the line with maximal P passes through the point of intersection of the two lines $y = 3$ and $y = -x + 5$. This means that $y = 3$, $x = 2$ and so $P_{max} = x + 2y = 8$.

A graphical method of solution is clearly limited to linear programming problems involving two variables only. However, it is a useful introduction to other techniques, so let us deal with another example.

Example 2

Maximise $\quad P = x + 4y$

subject to $\quad -x + 2y \leq 6$

$$5x + 4y \leq 40$$

$$x, y \geq 0$$

First of all, plot the appropriate straight line graphs to obtain the feasible polygon. This gives

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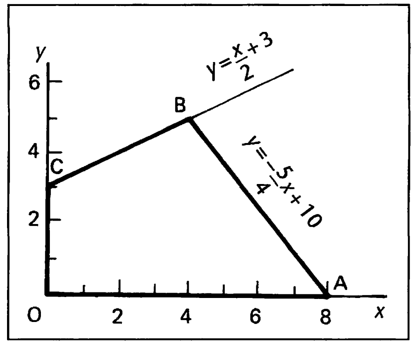

The objective function $P = x + 4y$ can be expressed in the form $y = -\frac{x}{4} + \frac{P}{4}$ and its graph added to the feasible polygon, as before. We then obtain

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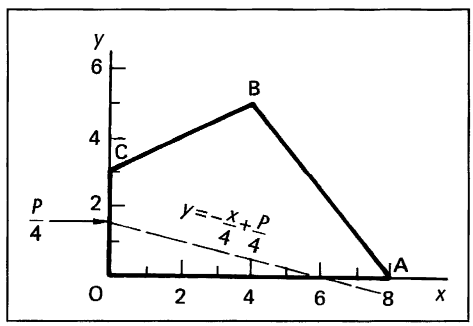

The line $y = -\frac{x}{4} + \frac{P}{4}$ is then raised to give the optimal solution, which is

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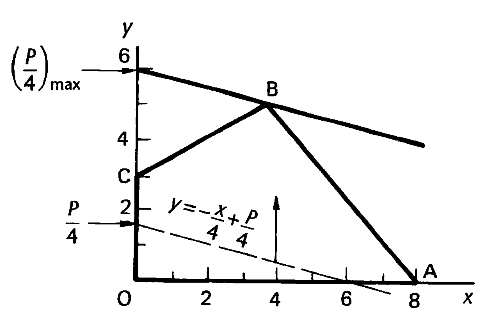
$$\boxed{P_{\text{max}} = 24 \text{ with } x = 4, y = 5}$$



From the graph it can be seen that the line with maximal P passes through the point of intersection of the two lines $y = \frac{x}{2} + 3$ and $y = -\frac{5x}{4} + 10$. That is $x = 4$, $y = 5$ and so $P_{\text{max}} = x + 4y = 24$.

As easy as that.

Now this one.

Example 3

Minimise $\quad P = -4x + 6y$

subject to $\quad -x + 6y \geq 24$

$2x - y \leq 7$

$x + 8y \leq 80$

$x, y \geq 0$

It is very much as before. Complete it on your own.

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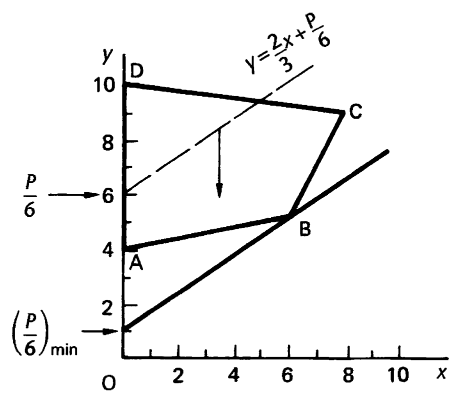
$$\boxed{P_{\text{min}} = 6 \quad \text{with} \quad x = 6,\ y = 5}$$



To obtain the minimum optimal value of P, the graph of the objective function is, of course, lowered to the appropriate extreme point.

In practice, linear programming problems usually contain many more variables than the two we have so far considered and a computational method is then required. One such technique is the *simplex method* and the remainder of this Programme will be devoted to the steps necessary to put it into practice.

So move on to Frame 14

The simplex method

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The first step in the *simplex method* is to ensure that each constraint is written with a *positive* right-hand side constant term. Then we express all inequalities as equations by the introduction of *slack variables*.

For example, $-x+2y \le 6$ can be written $-x+2y+w_1 = 6$
and $5x+4y \le 40$ can be written $5x+4y+w_2 = 40$

where w_1 and w_2 are positive (or zero) variables with unit coefficients, required to make up the left-hand side to the value of the right-hand side constant term. The new variables, w_1 and w_2, are called *slack variables*.

Let us look again at the problem we solved earlier.

Example 1

Maximise $P = x+4y$

subject to $-x+2y \le 6$

$5x+4y \le 40$ (as always, $x, y \ge 0$)

The constraints now become

$$-x+2y+w_1 = 6$$
$$5x+4y+w_2 = 40$$

and the objective function $P-x-4y = 0$

From this, we can now begin to form the simplex tableau (or table).

So make a note of the above information – and then move on

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Setting up the simplex tableau

(a) *Framework* First construct a framework with the headings shown.

x y w_1 w_2	b	*check*

Next, we enter, in the framework, the coefficients of the problem variables and of the slack variables in the constraints, together with the right-hand side constants in the column headed b. (Ignore the *check* column for the time being.)

So we have

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Problem variables | Slack variables | Const.

x	y	w_1	w_2	b	*check*
-1	2	1	0	6	
5	4	0	1	40	

body | unity matrix

(b) *Check column* The right-hand side column is included to provide a check on the numerical calculations as we develop the simplex, so, for each row, total up the entries in that row, including the constant column, and enter the sum in the check column.

Do that

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Basis	x	y	w_1	w_2	b	*check*
w_1	-1	2	1	0	6	8
w_2	5	4	0	1	40	50

(c) *Starting basic solution* The two constraints now contain four variables, but if we start by letting x and y each be zero, then we have the temporary solutions, $w_1 = 6$ and $w_2 = 40$, and we indicate these variables in the extra left-hand side column, as shown.

Note (1) The columns with the slack variables form a unity matrix.

(2) There are now four variables, x, y, w_1, w_2 $(n = 4)$.

(3) There are two constraints $(m = 2)$.

(4) We put $(n - m)$ variables, i.e. two variables (x and y), equal to zero as a start.

Finally, we have to deal with the objective function, so move on to the next frame

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(d) *The objective function* The objective function, $P = x + 4y$, is written $P - x - 4y = 0$ and the coefficients of this form the bottom row, or *index row*, of the tableau, thus

Basis	x	y	w_1	w_2	b	*check*
w_1	-1	2	1	0	6	8
w_2	5	4	0	1	40	50
P	-1	-4	0	0	0	-5

Complete your tableau, if you have not already done so, and then we will see how the computation is carried out.

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Computation of the simplex

1 First we select the column containing the most negative entry in the index row: in this case −4. This is called the *key column* and we enclose it as shown.

Basis	x	y	w_1	w_2	b	*check*
w_1	−1	2	1	0	6	8
w_2	5	4	0	1	40	50
P	−1	−4	0	0	0	−5

key column

2 In each row, we now divide the value in the b column by the positive entry in the key column: the smaller ratio determines the *key row*.

$$\left.\begin{aligned}\text{For row 1 } (w_1),\quad & r = \frac{6}{2} = 3 \\ \text{row 2 } (w_2),\quad & r = \frac{40}{4} = 10\end{aligned}\right\}$$

∴ row 1 is the key row. Enclose it as shown below.

Basis	x	y (pivot)	w_1	w_2	b	*check*	
w_1	−1	2	1	0	6	8	← key row
w_2	5	4	0	1	40	50	
P	−1	−4	0	0	0	−5	

The number at the intersection of the key column and key row is the *key number* or *pivot*: in this case the number 2.

3 We now divide all entries in the key row by the pivot to reduce the pivot to *unity* – which we then circle. The new version of the key row is sometimes called the *main row*. The rest of the tableau remains unchanged, so we then get ....

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Basis	x	y (unit pivot)	w_1	w_2	b	*check*	
w_1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4	← main row
w_2	5	4	0	1	40	50	
P	−1	−4	0	0	0	−5	← index

key column

So far, so good. Now we deal with the actual calculations.

Next frame

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4 Using the main row, we now operate on the remaining rows of the tableau, including the index row, to reduce the other entries in the key column to zero. Note that the main row remains unaltered. The new value in any position in the other rows, including the b column and the check column, can be calculated as follows:

New number = old number – the product of the corresponding entries in the main row and key column

1	A	main row
B	K	

K is replaced by $K - (A \times B)$

For example, in the second row (w_2):

5 is replaced by $5 - (-\frac{1}{2})(4) = 5 + 2 = 7$

4 is replaced by $4 - (1)(4) \quad = 4 - 4 = 0$

0 is replaced by $0 - (\frac{1}{2})(4) \quad = 0 - 2 = -2$

1 is replaced by $1 - (0)(4) \quad = 1 - 0 = 1$

40 is replaced by $40 - (3)(4) \quad = 40 - 12 = 28$

50 is replaced by $50 - (4)(4) \quad = 50 - 16 = 34$

and, in the third row (P):

-1 is replaced by $-1 - (-\frac{1}{2})(-4) = -1 - 2 = -3.$

Completing the operations for rows (w_2) and (P), we have

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Basis	x	y	w_1	w_2	b	*check*
w_1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	7	0	-2	1	28	34
P	-3	0	2	0	12	11

Now confirm that the new values in the check column are, indeed, the sums of the entries in the corresponding rows. If not, there is a mistake somewhere in the working to be corrected before we proceed.

If all is well, move on to the next frame

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5 *Change of basic variables* In its final form, the key column consists of a single 1 and the remaining entries zero. This is in the column headed y which indicates that the basic variable w_1 in the main row can be replaced by y.

Basis	x	y	w_1	w_2	b	check
y ~~w_1~~	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	7	0	-2	1	28	34
P	-3	0	2	0	12	11

Note that there are two columns containing a single 1 and the rest 0. These are headed y and w_2 which are now also the basic variables in the left-hand side column. Reading the values in the b column therefore gives a basic solution $y = 3$ and $w_2 = 28$, and at this stage $P = 12$. Any variable not listed in the basis column is zero. One basic solution at this stage is therefore $x = 0$, $y = 3$, $w_2 = 28$. However, we are not finished.

The index row (P) still contains another negative entry, so we have to repeat the simplex process using the same steps as before.

Basis	x	y	w_1	w_2	b	check	
y	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4	
w_2	7	0	-2	1	28	34	← key row
P	-3	0	2	0	12	11	
	↑ key column						

Now divide the key row by the key number (7) to reduce the pivot to a unit pivot. This gives

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Basis	x	y	w_1	w_2	b	check	
y	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4	
w_2	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$	← main row
P	-3	0	2	0	12	11	

Using the main row, operate on the remaining rows (including the index row) to reduce the other entries in the key column to zero. Complete that stage and we have

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Basis	x	y	w_1	w_2	b	check
y	0	1	$\frac{5}{14}$	$\frac{1}{14}$	5	$\frac{45}{7}$
w_2	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	0	0	$\frac{8}{7}$	$\frac{3}{7}$	24	$\frac{179}{7}$

Again, at this stage, check your working by totalling up the entries in each row and satisfy yourself that the sum agrees with the value in the check column.

Note that w_2 in the basis column can now be replaced by x which was the heading of the column containing the last unit pivot.

So finally, we have

Basis	x	y	w_1	w_2	b	check
y	0	1	$\frac{5}{14}$	$\frac{1}{14}$	5	$\frac{45}{7}$
x	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	0	0	$\frac{8}{7}$	$\frac{3}{7}$	24	$\frac{179}{7}$

A new basic solution now emerges as $x = 4$, $y = 5$.

Furthermore, since there is no further negative entry in the index row, this is also the optimal solution and the optimal value of P is given in the b column, i.e. $P_{\text{max}} = 24$.

For interest, you may wish to compare this result with that obtained in Frame 12.

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We have been through the simplex operation in some detail by way of explanation. Many problems involve more than just two variables, but the method of computation is basically the same, being an iterative process which is repeated until the index row contains no negative entry, at which point the optimal value of the objective function has been attained.

The problem we have just solved would normally look like this:

$$\begin{aligned} &\text{Maximise} \quad && P = x + 4y \\ &\text{subject to} && -x + 2y \le 6 \\ & && 5x + 4y \le 40 \quad (x,\ y \ge 0) \end{aligned}$$

Entering slack variables, etc., this is written

$$\begin{aligned} -x + 2y + w_1 \qquad &= 6 \\ 5x + 4y \qquad + w_2 &= 40 \\ P - x - 4y \qquad &= 0 \end{aligned}$$

The complete tableau is given in the next frame

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Basis	x	y	w_1	w_2	b	*check*
w_1	-1	2	1	0	6	8
w_2	5	4	0	1	40	50
P	-1	-4	0	0	0	-5
$y\ \cancel{w_1}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	5	4	0	1	40	50
P	-1	-4	0	0	0	-5
$y\ \cancel{w_1}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	7	0	-2	1	28	34
P	-3	0	2	0	12	11
y	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	-3	0	2	0	12	11
y	0	1	$\frac{5}{14}$	$\frac{1}{14}$	5	$\frac{45}{7}$
$x\ \cancel{w_2}$	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	0	0	$\frac{8}{7}$	$\frac{3}{7}$	24	$\frac{179}{7}$

$$P_{\max} = 24 \text{ with } x = 4,\ y = 5$$

Now for another example – so move on to Frame 28

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Here is one for you to do on your own. The method is just the same as before so you will have no difficulty.

Example 2

$$\begin{aligned} \text{Maximise} \quad & P = 4x + 3y \\ \text{subject to} \quad & -x + \ \ y \le 4 \\ & \ \ x + 2y \le 14 \\ & 2x + \ \ y \le 16 \quad (x, y \ge 0) \end{aligned}$$

We have three inequalities this time, so we shall need to introduce three slack variables. Converting the inequalities into equations, we obtain

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$$\begin{aligned} -x + \ y + w_1 \qquad\qquad &= 4 \\ x + 2y \qquad + w_2 \qquad &= 14 \\ 2x + \ y \qquad\qquad + w_3 &= 16 \end{aligned}$$

Then we set out the simplex framework with appropriate headings, i.e.

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Basis	x y w_1 w_2 w_3	b	*check*

Remembering that the index row uses $P - 4x - 3y = 0$, we can set out the first tableau. Choosing x and y, as usual, to be zero for a start, we have

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Basis	x	y	w_1	w_2	w_3	b	*check*
w_1	-1	1	1	0	0	4	5
w_2	1	2	0	1	0	14	18
w_3	2	1	0	0	1	16	20
P	-4	-3	0	0	0	0	-7

Carry on now and complete the working on this first tableau.

Check with the next frame

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Here is the working so far.

Basis	x	y	w_1	w_2	w_3	b	check
w_1	-1	1	1	0	0	4	5
w_2	1	2	0	1	0	14	18
w_3	2	1	0	0	1	16	20
P	-4	-3	0	0	0	0	-7
w_1	-1	1	1	0	0	4	5
w_2	1	2	0	1	0	14	18
w_3	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	-4	-3	0	0	0	0	-7
w_1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	15
w_2	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	6	8
x $\cancel{w_3}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	0	-1	0	0	2	32	33

(a) The basic variable (w_3) of the unit pivot can now be replaced by the variable at the heading of the unit pivot (x).

(b) We see there is still a negative value in the index row, so we repeat the process for this second tableau.

Now you can finish it off

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Check to see if you agree.

Basis	x	y	w_1	w_2	w_3	b	*check*
w_1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	15
w_2	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	6	8
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	0	-1	0	0	2	32	33
w_1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	15
w_2	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	4	$\frac{16}{3}$
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	0	-1	0	0	2	32	33
w_1	0	0	1	-1	1	6	7
y $\cancel{w_2}$	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	4	$\frac{16}{3}$
x	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	6	$\frac{22}{3}$
P	0	0	0	$\frac{2}{3}$	$\frac{5}{3}$	36	$\frac{115}{3}$

The basic variable (w_2) can now be replaced by y, being the heading of the unit pivot column.

There is no further negative entry in the index row: therefore, the optimal value of P has been attained.

$\therefore\ P_{\max} = 36 \quad \text{with} \quad x = 6,\ y = 4.$

Note: We also see that $w_1 = 6$, since the unity matrix has headings x, y, w_1. The full result, therefore, is

$P_{\max} = 36 \quad \text{with} \quad x = 6, y = 4, w_1 = 6, w_2 = 0, w_3 = 0$

though we do not normally require this extra information.

The meaning of $w_1 = 0$ and $w_2 = 0$ is that the second and third constraints become

$x + 2y = 14$ and $2x + y = 16$ respectively rather than
$x + 2y \leq 14$ and $2x + y \leq 16$.

The meaning of $w_1 = 6$ gives the first constraint as $-x + y < 14$ rather than $-x + y \leq 14$.

Now we will extend the simplex method to an example involving three problem variables.

Next frame

34 Simplex with three problem variables

Maximise $P = p_1x + p_2y + p_3z$

subject to $a_{11}x + a_{12}y + a_{13}z \leq b_1$

$a_{21}x + a_{22}y + a_{23}z \leq b_2$

$a_{31}x + a_{32}y + a_{33}z \leq b_3$

$x, y, z \geq 0$

Introducing slack variables we have

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z + w_1 \qquad\qquad &= b_1 \\ a_{21}x + a_{22}y + a_{23}z \qquad + w_2 \qquad &= b_2 \\ a_{31}x + a_{32}y + a_{33}z \qquad\qquad + w_3 &= b_3 \\ P - p_1x - p_2y - p_3z \qquad\qquad\qquad &= 0 \end{aligned}$$

If there is now a total of n variables and m constraints, then at least $(n - m)$ variables are equated to zero. The remainder form the basic variable column entries. Equating x, y, z to zero, then, the basic variables are w_1, w_2, w_3.

Basis	x	y	z	w_1	w_2	w_3	b	*check*
w_1	a_{11}	a_{12}	a_{13}	1	0	0	b_1	
w_2	a_{21}	a_{22}	a_{23}	0	1	0	b_2	
w_3	a_{31}	a_{32}	a_{33}	0	0	1	b_3	
P	$-p_1$	$-p_2$	$-p_3$	0	0	0	0	

The variables in the basis column are the variables heading the unity matrix. The method is exactly as before.

(a) Select the most negative entry in the index row to determine the *key column*.

(b) Divide the entries in the constant column (b) by the corresponding positive entries in the key column. The smallest positive ratio determines the *key row*.

(c) The entry at the intersection of the key column and the key row is the *key number* or *pivot*.

(d) Divide each entry in the key row by the pivot to reduce the key number to a *unit pivot*. The revised key row is now called the *main row*.

(e) Use the main row to operate on the remaining rows to reduce all other entries in the key column to zero.

(f) Repeat steps (a) to (e) until no negative entry remains in the index row.

Now for an example

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Example 1

Maximise $P = 2x + 6y + 4z$

subject to

$$\begin{aligned} 2x + 5y + 2z &\leq 38 \\ 4x + 2y + 3z &\leq 57 \\ x + 3y + 5z &\leq 57 \\ x, y, z &\geq 0 \end{aligned}$$

Rewriting the inequalities as equations gives

....

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$$\boxed{\begin{aligned} 2x + 5y + 2z + w_1 \qquad\qquad &= 38 \\ 4x + 2y + 3z \qquad + w_2 \qquad &= 57 \\ x + 3y + 5z \qquad\qquad + w_3 &= 57 \end{aligned}}$$

We also have $P - 2x - 6y - 4z = 0$, so we can now set up the simplex tableau ready for solution. That is ....

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Basis	x	y	z	w_1	w_2	w_3	b	*check*
w_1	2	5	2	1	0	0	38	48
w_2	4	2	3	0	1	0	57	67
w_3	1	3	5	0	0	1	57	67
P	−2	−6	−4	0	0	0	0	− 12

Now we just apply the normal simplex routine until there is no negative entry in the index row.

Remember:

(1) to replace the basic variables as the problem variables become available at each stage, and

(2) any variable not appearing in the basis column has zero value.

Now you can work the solution right through and then check the result with the next frame. Take your time: there are no snags.

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Basis	x	y	z	w_1	w_2	w_3	b	check
w_1	2	5	2	1	0	0	38	48
w_2	4	2	3	0	1	0	57	67
w_3	1	3	5	0	0	1	57	67
P	-2	-6	-4	0	0	0	0	-12
w_1	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{38}{5}$	$\frac{48}{5}$
w_2	4	2	3	0	1	0	57	67
w_3	1	3	5	0	0	1	57	67
P	-2	-6	-4	0	0	0	0	-12
y $\cancel{w_1}$	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{38}{5}$	$\frac{48}{5}$
w_2	$\frac{16}{5}$	0	$\frac{11}{5}$	$-\frac{2}{5}$	1	0	$\frac{209}{5}$	$\frac{239}{5}$
w_3	$-\frac{1}{5}$	0	$\frac{19}{5}$	$-\frac{3}{5}$	0	1	$\frac{171}{5}$	$\frac{191}{5}$
P	$\frac{2}{5}$	0	$-\frac{8}{5}$	$\frac{6}{5}$	0	0	$\frac{228}{5}$	$\frac{228}{5}$
y	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{38}{5}$	$\frac{48}{5}$
w_2	$\frac{16}{5}$	0	$\frac{11}{5}$	$-\frac{2}{5}$	1	0	$\frac{209}{5}$	$\frac{239}{5}$
w_3	$-\frac{1}{19}$	0	1	$-\frac{3}{19}$	0	$\frac{5}{19}$	9	$\frac{191}{19}$
P	$\frac{2}{5}$	0	$-\frac{8}{5}$	$\frac{6}{5}$	0	0	$\frac{228}{5}$	$\frac{228}{5}$
y	$\frac{8}{19}$	1	0	$\frac{5}{19}$	0	$-\frac{2}{19}$	4	$\frac{106}{19}$
w_2	$\frac{63}{19}$	0	0	$-\frac{1}{19}$	1	$-\frac{11}{19}$	22	$\frac{448}{19}$
z $\cancel{w_3}$	$-\frac{1}{19}$	0	1	$-\frac{3}{19}$	0	$\frac{5}{19}$	9	$\frac{191}{19}$
P	$\frac{16}{19}$	0	0	$\frac{18}{19}$	0	$\frac{8}{19}$	60	$\frac{1172}{19}$

$$\therefore\ P_{\max} = 60 \text{ with } x = 0,\ y = 4,\ z = 9.$$

Do you agree?

If so, on to the next frame

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Example 2

Maximise $P = 3x + 4y + 5z$

subject to

$$2x + 4y + 3z \le 80$$
$$4x + 2y + z \le 48$$
$$x + y + 2z \le 40$$
$$x, y, z \ge 0$$

It is much the same as before. Work through it carefully and then check with the next frame.

40

$$
\begin{aligned}
2x + 4y + 3z + w_1 \qquad\qquad &= 80\\
4x + 2y + \ z \qquad + w_2 \qquad &= 48\\
x + \ y + 2z \qquad\qquad + w_3 &= 40\\
P - 3x - 4y - 5z \qquad\qquad &= 0
\end{aligned}
$$

Basis	x	y	z	w_1	w_2	w_3	b	*check*
w_1	2	4	3	1	0	0	80	90
w_2	4	2	1	0	1	0	48	56
w_3	1	1	2	0	0	1	40	45
P	−3	−4	−5	0	0	0	0	−12
w_1	2	4	3	1	0	0	80	90
w_2	4	2	1	0	1	0	48	56
w_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	20	$\frac{45}{2}$
P	−3	−4	−5	0	0	0	0	−12
w_1	$\frac{1}{2}$	$\frac{5}{2}$	0	1	0	$-\frac{3}{2}$	20	$\frac{45}{2}$
w_2	$\frac{7}{2}$	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	28	$\frac{67}{2}$
$z\ \cancel{w_3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	20	$\frac{45}{2}$
P	$-\frac{1}{2}$	$-\frac{3}{2}$	0	0	0	$\frac{5}{2}$	100	$\frac{201}{2}$
w_1	$\frac{1}{5}$	1	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	8	9
w_2	$\frac{7}{2}$	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	28	$\frac{67}{2}$
z	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	20	$\frac{45}{2}$
P	$-\frac{1}{2}$	$-\frac{3}{2}$	0	0	0	$\frac{5}{2}$	100	$\frac{201}{2}$
$y\ \cancel{w_1}$	$\frac{1}{5}$	1	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	8	9
w_2	$\frac{16}{5}$	0	0	$-\frac{3}{5}$	1	$\frac{2}{5}$	16	20
z	$\frac{2}{5}$	0	1	$-\frac{1}{5}$	0	$\frac{4}{5}$	16	18
P	$-\frac{1}{5}$	0	0	$\frac{3}{5}$	0	$\frac{8}{5}$	112	114
y	$\frac{1}{5}$	1	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	8	9
w_2	1	0	0	$-\frac{3}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	5	$\frac{25}{4}$
z	$\frac{2}{5}$	0	1	$-\frac{1}{5}$	0	$\frac{4}{5}$	16	18
P	$-\frac{1}{5}$	0	0	$\frac{3}{5}$	0	$\frac{8}{5}$	112	114
y	0	1	0	$\frac{7}{16}$	$-\frac{1}{16}$	$-\frac{5}{8}$	7	$\frac{31}{4}$
$x\ \cancel{w_2}$	1	0	0	$-\frac{3}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	5	$\frac{25}{4}$
z	0	0	1	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{4}$	14	$\frac{31}{2}$
P	0	0	0	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{13}{8}$	113	$\frac{461}{4}$

$$\therefore\ P_{\max} = 113 \quad \text{with } x = 5,\ y = 7,\ z = 14.$$

Now let us meet a further complication. *Next frame*

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Artificial variables

So far, our approach to each problem has been the same.

(a) We first of all convert the 'less than' inequalities into equations by the inclusion of slack variables.

(b) If there are now n variables and m constraints, then at least $(n-m)$ variables are equated to zero – usually x, y, z, etc. – so that the initial basic solution is given by the slack variables, the coefficients of which form the unity matrix in the tableau.

(c) We then proceed by the simplex method to convert the basic solution to one containing the problem variables, the tableau entries for which now form a new unity matrix.

(d) The method is repeated as necessary. When no negative entry remains in the index row, the value of P denoted in the constant column is the optimal value of the objective function.

Now let us look at this example.

Example 1

Maximise $\quad P = 7x + 4y$

subject to
$$\begin{aligned} 2x + y &\leq 150 \\ 4x + 3y &\leq 350 \\ x + y &\geq 80 \quad (x, y \geq 0) \end{aligned}$$

Converting the inequalities to equations, we have

....

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$$\boxed{\begin{aligned} 2x + y + w_1 \qquad\qquad &= 150 \\ 4x + 3y \qquad + w_2 \qquad &= 350 \\ x + y \qquad\qquad - w_3 &= 80 \end{aligned}}$$

Also, of course, $P - 7x - 4y = 0$.

NOTE that since the third constraint is a 'greater than' statement, we must subtract the slack positive variable (w_3) to form the equation.

Alternatively, we could have written the inequality as $-x - y \leq -80$ so that $-x - y + w_3 = -80$ and so $x + y - w_3 = 80$.

Forming the first tableau, in the usual manner, we obtain

....

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Basis	x	y	w_1	w_2	w_3	b	*check*
w_1	2	1	1	0	0	150	154
w_2	4	3	0	1	0	350	358
w_3	1	1	0	0	−1	80	81
P	−7	−4	0	0	0	0	−11

Now we are stuck, for we do not have a unity matrix to start off with. The entry in the w_3 column is –1 and not the necessary +1, and no amount of manipulation will help since the entries in the constant column (b) are, by definition, positive.

So how can we find a starting technique?

Let us restate the problem.

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Maximise $P = 7x + 4y$

subject to

$$\begin{aligned} 2x + y + w_1 \quad\quad\quad &= 150 \\ 4x + 3y \quad + w_2 \quad &= 350 \\ x + y \quad\quad - w_3 &= 80 \end{aligned}$$

The trouble comes in the third constraint by virtue of the negative sign of the slack variable. To save the situation, we introduce a new small positive variable (w_4) so that w_1, w_2 and w_4 will give rise to a unity matrix and the simplex computation can then proceed. Of course, w_4 is ficticious, is extremely small and cannot appear in the final basic solution. To establish this, we include in the objective function a new term $-Mw_4$, where M is an extremely large positive value which will ensure that w_4 will ultimately vanish. So we now write

$$P = 7x + 4y - Mw_4$$

The new variable, w_4, is called an *artificial variable*: it is introduced solely so that the simplex procedure can be carried out; and it must not appear in the final basic solution listed in the basis column.

The third constraint above now becomes

$$x + y \quad\quad - w_3 + w_4 = 80$$

Next frame

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We now have

$$\begin{aligned} 2x + \ y + w_1 \qquad\qquad\qquad\qquad &= 150 \\ 4x + 3y \qquad + w_2 \qquad\qquad\quad &= 350 \\ x + \ y \qquad\qquad - w_3 + \quad w_4 &= \ 80 \\ P - 7x - 4y \qquad\qquad\qquad + Mw_4 &= \ 0 \end{aligned}$$

Forming the tableau in the usual way:

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
w_1	2	1	1	0	0	0	150	154
w_2	4	3	0	1	0	0	350	358
w_4	1	1	0	0	−1	1	80	82
P	−7	−4	0	0	0	M	0	$M - 11$

Note that:

(1) The columns headed w_1, w_2, w_4 now form the unity matrix.
(2) There are now 6 variables and 3 constraints, i.e. $n = 6$ and $m = 3$. At least $(n - m)$, i.e. $6 - 3 = 3$, variables are put equal to zero. We start off with x, y, w_3 as zero and w_1, w_2, w_4 form the first basic solution with the values given in the b column.

We now proceed in the normal way. Solve the first tableau and check the results so far in the following frame.

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Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	2	1	1	0	0	0	150	154
w_2	4	3	0	1	0	0	350	358
w_4	1	1	0	0	-1	1	80	82
P	-7	-4	0	0	0	M	0	$M-11$
w_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	4	3	0	1	0	0	350	358
w_4	1	1	0	0	-1	1	80	82
P	-7	-4	0	0	0	M	0	$M-11$
x ~~w_1~~	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	0	1	-2	1	0	0	50	50
w_4	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1	5	5
P	0	$-\frac{1}{2}$	$\frac{7}{2}$	0	0	M	525	$M+528$

The basic variable w_1 can be replaced by x and we continue as before to remove the further negative entry in the index row. *Do that*

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Basis	x	y	w_1	w_2	w_3	w_4	b	check
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	0	1	-2	1	0	0	50	50
w_4	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1	5	5
P	0	$-\frac{1}{2}$	$\frac{7}{2}$	0	0	M	525	$M+528$
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	0	1	-2	1	0	0	50	50
w_4	0	1	-1	0	-2	2	10	10
P	0	$-\frac{1}{2}$	$\frac{7}{2}$	0	0	M	525	$M+528$
x	1	0	1	0	1	-1	70	72
w_2	0	0	-1	1	2	-2	40	40
y ~~w_4~~	0	1	-1	0	-2	2	10	10
P	0	0	3	0	-1	$M+1$	530	$M+533$

The basic variable w_4 is now replaced by y (the column of the last unit pivot). We now have another negative entry in the index row, so we have to perform the simplex calculation yet again.

For the next round, we get ....

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Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
x	1	0	1	0	1	-1	70	72
w_2	0	0	-1	1	2	-2	40	40
y	0	1	-1	0	-2	2	10	10
P	0	0	3	0	-1	$M+1$	530	$M+533$
x	1	0	1	0	1	-1	70	72
w_2	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	20	20
y	0	1	-1	0	-2	2	10	10
P	0	0	3	0	-1	$M+1$	530	$M+533$
x	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	50	52
$w_3\ \cancel{w_2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	20	20
y	0	1	-2	1	0	0	50	50
P	0	0	$\frac{5}{2}$	$\frac{1}{2}$	0	M	550	$M+553$

No further negative entry remains in the index row. The optimal solution has been found, i.e.

$P_{\max} = 550$ with $x = 50$, $y = 50$.

In addition, we see that $w_3 = 20$ while $w_1 = w_2 = w_4 = 0$ since they do not occur in the basic variable column.

Notice, also, that w_4, the artificial variable, does not figure in the optimal solution – as indeed it must not.

Next frame

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Here is one for you to do in the same way.

Example 2

Maximise $\quad P = 2x + 5y$

subject to $\quad x + 4y \leq 60$

$$3x + 2y \leq 40$$

$$x + y \geq 12 \quad (x, y \geq 0)$$

Work right through it, just as before. The result is

....

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$$\boxed{P_{\max} = 78 \quad \text{with} \quad x = 4,\ y = 14}$$

Because

$$\begin{aligned}
x + 4y + w_1 \quad & = 60 \\
3x + 2y \quad + w_2 \quad & = 40 \\
x + y \quad - w_3 + w_4 & = 12 \\
P - 2x - 5y \quad + Mw_4 & = 0
\end{aligned}$$

Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	1	4	1	0	0	0	60	66
w_2	3	2	0	1	0	0	40	46
w_4	1	1	0	0	−1	1	12	14
P	−2	−5	0	0	0	M	0	$M-7$
w_1	−3	0	1	0	4	−4	12	10
w_2	1	0	0	1	2	−2	16	18
y ~~w_4~~	1	1	0	0	−1	1	12	14
P	3	0	0	0	−5	$M+5$	60	$M+63$
w_1	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	1	−1	3	$\frac{5}{2}$
w_2	1	0	0	1	2	−2	16	18
y	1	1	0	0	−1	1	12	14
P	3	0	0	0	−5	$M+5$	60	$M+63$
w_3 ~~w_1~~	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	1	−1	3	$\frac{5}{2}$
w_2	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	0	0	10	13
y	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	0	15	$\frac{33}{2}$
P	$-\frac{3}{4}$	0	$\frac{5}{4}$	0	0	M	75	$M+\frac{151}{2}$
w_3	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	1	−1	3	$\frac{5}{2}$
w_2	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	0	4	$\frac{26}{2}$
y	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	0	15	$\frac{33}{2}$
P	$-\frac{3}{4}$	0	$\frac{5}{4}$	0	0	M	75	$M+\frac{151}{2}$
w_3	0	0	$\frac{1}{10}$	$\frac{3}{10}$	1	−1	6	$\frac{32}{5}$
x ~~w_2~~	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	0	4	$\frac{26}{5}$
y	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	0	0	14	$\frac{76}{5}$
P	0	0	$\frac{11}{10}$	$\frac{3}{10}$	0	M	78	$M+\frac{397}{5}$

$\therefore\ P_{\max} = 78$ with $x = 4,\ y = 14$.

On to the next frame

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We are not always as lucky as we were in the previous two examples and other steps sometimes have to be taken to remove the artificial variable. Consider the following case.

Example 3

$$\begin{array}{ll} \text{Maximise} & P = 8x + 4y \\ \text{subject to} & 2x + 3y \leq 120 \\ & x + \ \ y \leq \ \ 45 \\ & -3x + 5y \geq \ \ 25 \quad (x, y \geq 0) \end{array}$$

Inserting the slack variables and artificial variable as required, we have

....

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$$\boxed{\begin{array}{rrrrr} 2x + 3y + w_1 & & & & = 120 \\ x + \ \ y & + w_2 & & & = \ \ 45 \\ -3x + 5y & & - w_3 + & w_4 = & 25 \\ P - 8x - 4y & & & + Mw_4 = & 0 \end{array}}$$

That is very much as before, so work through it and check with the next frame.

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Here it is.

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
w_1	2	3	1	0	0	0	120	126
w_2	1	1	0	1	0	0	45	48
w_4	−3	5	0	0	−1	1	25	27
P	−8	−4	0	0	0	M	0	$M - 12$
w_1	0	1	1	−2	0	0	30	30
x $\cancel{w_2}$	1	1	0	1	0	0	45	48
$* \rightarrow w_4$	0	8	0	3	−1	1	160	171
P	0	4	0	8	0	M	360	$M + 372$

There is no further negative entry in the index row, so it looks as though the optimal solution has been attained. However, the artificial variable w_4 still remains in the basic variable column at * and thus must be removed. Therefore, we take the entry at the junction of the y column and the w_4 row as the pivot and proceed to eliminate w_4 by simplifying the tableau a stage further.

If we do that, we get ....

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Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	0	1	1	-2	0	0	30	30
x	1	1	0	1	0	0	45	48
$* \rightarrow w_4$	0	$\boxed{8}$	0	3	-1	1	160	171
P	0	4	0	8	0	M	360	$M+372$
w_1	0	1	1	-2	0	0	30	30
x	1	1	0	1	0	0	45	48
$* \rightarrow w_4$	0	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	20	$\frac{171}{8}$
P	0	4	0	8	0	M	360	$M+372$
w_1	0	0	1	$-\frac{19}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	10	$\frac{69}{8}$
x	1	0	0	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	25	$\frac{213}{8}$
y ~~w_4~~	0	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	50	$\frac{171}{8}$
P	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	$M-\frac{1}{2}$	280	$M+\frac{573}{2}$

The artificial variable, w_4, is now replaced by y in the basic variable column and the optimal solution has been reached.

$\therefore\; P_{max} = 280 \quad \text{with} \quad x = 25,\; y = 20.$

Next frame

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Now here is one for you to deal with.

Example 4

Maximise $P = 10x + 2y$

subject to
$$-x + 2y \leq 60$$
$$5x + 4y \leq 260$$
$$-x + 8y \geq 80 \quad (x, y \geq 0)$$

Work through it as before and see if you agree with the solution in the next frame.

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$$\begin{aligned}
-x + 2y + w_1 \qquad\qquad\qquad &= 60 \\
5x + 4y \qquad + w_2 \qquad\qquad &= 260 \\
-x + 8y \qquad\qquad - w_3 + \quad w_4 &= 80 \\
P - 10x - 2y \qquad\qquad\qquad + Mw_4 &= 0
\end{aligned}$$

Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	-1	2	1	0	0	0	60	62
w_2	5	4	0	1	0	0	260	270
w_4	-1	8	0	0	-1	1	80	87
P	-10	-2	0	0	0	M	0	$M-12$
w_1	-1	2	1	0	0	0	60	62
w_2	1	$\frac{4}{5}$	0	$\frac{1}{5}$	0	0	52	54
w_4	-1	8	0	0	-1	1	80	87
P	-10	-2	0	0	0	M	0	$M-12$
w_1	0	$\frac{14}{5}$	1	$\frac{1}{5}$	0	0	112	116
x ~~w_2~~	1	$\frac{4}{5}$	0	$\frac{1}{5}$	0	0	52	54
* → w_4	0	$\boxed{\frac{44}{5}}$	0	$\frac{1}{5}$	-1	1	132	141
P	0	6	0	2	0	M	520	$M+528$
w_1	0	$\frac{14}{5}$	1	$\frac{1}{5}$	0	0	112	116
x	1	$\frac{4}{5}$	0	$\frac{1}{5}$	0	0	52	54
* → w_4	0	1	0	$\frac{1}{44}$	$-\frac{5}{44}$	$\frac{5}{44}$	15	$\frac{705}{44}$
P	0	6	0	2	0	M	520	$M+528$
w_1	0	0	1	$\frac{3}{22}$	$\frac{7}{22}$	$-\frac{7}{22}$	70	$\frac{1565}{22}$
x	1	0	0	$\frac{2}{11}$	$\frac{1}{11}$	$-\frac{1}{11}$	40	$\frac{453}{11}$
y ~~w_4~~	0	1	0	$\frac{1}{44}$	$-\frac{5}{44}$	$\frac{5}{44}$	15	$\frac{705}{44}$
P	0	0	0	$\frac{41}{22}$	$\frac{15}{22}$	$M-\frac{15}{22}$	430	$M+\frac{9501}{22}$

$\therefore\ P_{max} = 430$ with $x = 40,\ y = 15.$

Now for another example

Example 5

This one is slightly different, so take note.

$$\begin{aligned} \text{Maximise} \quad & P = 11x + 15y \\ \text{subject to} \quad & 3x + 5y \leq 130 \\ & -4x + 5y \geq 25 \\ & x + 5y \geq 75 \quad (x, y \geq 0) \end{aligned}$$

In this problem, notice that there are two 'greater than' inequalities so that there will be two slack variables to be subtracted and two artificial variables to be incorporated. In the objective function, we can use the same factor, M, for both artificial variables, since neither of those two variables will appear in the final optimal solution. So, we have:

$$\begin{aligned} 3x + 5y + w_1 &= 130 \\ -4x + 5y - w_2 + w_4 &= 25 \\ x + 5y - w_3 + w_5 &= 75 \\ P - 11x - 15y + Mw_4 + Mw_5 &= 0 \end{aligned}$$

w_1, w_4, w_5 now form the unity matrix from which to start. The method is just the same as in previous examples, so finish it off.

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Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	check
w_1	3	5	1	0	0	0	0	130	139
w_4	−4	5	0	−1	0	1	0	25	26
w_5	1	5	0	0	−1	0	1	75	81
P	−11	−15	0	0	0	M	M	0	$2M-26$
w_1	3	5	1	0	0	0	0	130	139
w_4	$-\frac{4}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	5	$\frac{26}{5}$
w_5	1	5	0	0	−1	0	1	75	81
P	−11	−15	0	0	0	M	M	0	$2M-26$
w_1	7	0	1	1	0	−1	0	105	113
$y\ \cancel{w_4}$	$-\frac{4}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	5	$\frac{26}{5}$
w_5	5	0	0	1	−1	−1	1	50	55
P	−23	0	0	−3	0	$M+3$	M	75	$2M+52$
w_1	7	0	1	1	0	−1	0	105	113
y	$-\frac{4}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	5	$\frac{26}{5}$
w_5	1	0	0	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	10	11
P	−23	0	0	−3	0	$M+3$	M	75	$2M+52$
w_1	0	0	1	$-\frac{2}{5}$	$\frac{7}{5}$	$\frac{2}{5}$	$-\frac{7}{5}$	35	36
y	0	1	0	$-\frac{1}{25}$	$-\frac{4}{25}$	$\frac{1}{25}$	$\frac{4}{25}$	13	14
$x\ \cancel{w_5}$	1	0	0	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	10	11
P	0	0	0	$\frac{8}{5}$	$-\frac{23}{5}$	$M-\frac{8}{5}$	$M+\frac{23}{5}$	305	$2M+305$
w_1	0	0	$\frac{5}{7}$	$-\frac{2}{7}$	1	$\frac{2}{7}$	−1	25	$\frac{180}{7}$
y	0	1	0	$-\frac{1}{25}$	$-\frac{4}{25}$	$\frac{1}{25}$	$\frac{4}{25}$	13	14
x	1	0	0	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	10	11
P	0	0	0	$\frac{8}{5}$	$-\frac{23}{5}$	$M-\frac{8}{5}$	$M+\frac{23}{5}$	305	$2M+305$
$w_3\ \cancel{w_1}$	0	0	$\frac{5}{7}$	$-\frac{2}{7}$	1	$\frac{2}{7}$	−1	25	$\frac{180}{7}$
y	0	1	$\frac{4}{35}$	$-\frac{3}{35}$	0	$\frac{3}{35}$	0	17	$\frac{634}{35}$
x	1	0	$\frac{1}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	0	15	$\frac{113}{7}$
P	0	0	$\frac{23}{7}$	$\frac{2}{7}$	0	$M-\frac{2}{7}$	M	420	$2M+\frac{2963}{7}$

So there it is. $P_{\max}=420$ with $x=15$, $y=17$.

Incidentally, also, $w_3=25$ and $w_1=w_2=w_4=w_5=0$.

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Our examples on the use of artificial variables have so far concerned only two problem variables, x and y. The method, however, is exactly the same when more problem variables are involved, though, naturally, the solution then becomes somewhat longer.

Here is one for you to work through: it brings in most of what we have covered and provides excellent revision. The result is given in the next frame.

Example 6

$$\begin{array}{ll} \text{Maximise} & P = 24x + 21y + 30z \\ \text{subject to} & 12x + 4y + 8z \leq 240 \\ & 8x + 3y + 3z \leq 140 \\ & 6x + 2y + 3z \geq 110 \quad (x, y, z \geq 0) \end{array}$$

60

$$\boxed{P_{max} = 750 \text{ with } x = 10, y = 10, z = 10}$$

The simplex technique is designed to maximise a given objective function in the light of stated constraints. However, a problem requiring the minimisation of an objective function (denoting costs, machine idling time, etc.) can easily be converted for solution by the same method.

For this, move on to the next frame

Minimisation

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If P denotes the objective function to be minimised, we write Q as the negative of this function. Q_{max} is then determined by the usual simplex method and, finally, the negative value of Q_{max} is the value of the required P_{min}.

i.e. Write $Q = -P$. Determine Q_{max} in the normal way.

Then $P_{min} = -(Q_{max})$.

Example 1

$$\begin{array}{ll} \text{Minimise} & P = -3x + 4y \\ \text{subject to} & x + 3y \leq 54 \\ & 3x + y \leq 34 \\ & -x + 2y \geq 12 \quad (x, y \geq 0) \end{array}$$

First write $Q = -P$, i.e. $Q = 3x - 4y$, and maximise Q.

Inserting the usual slack variables and artificial variable as needed, we have ....

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$$\begin{array}{llll} x+3y+w_1 & & & =54 \\ 3x+\ \ y & +w_2 & & =34 \\ -x+2y & & -w_3+\ \ w_4 & =12 \\ Q-3x+4y & & +Mw_4 & =\ \ 0 \end{array}$$

Now we just carry out the usual simplex routine to evaluate Q_{max} and hence P_{min}, since $P_{min} = -(Q_{max})$. $P_{min} = \ldots\ldots\ldots\ldots$

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$$\boxed{P_{min} = 16 \quad \text{with} \quad x = 8, y = 10}$$

Because $Q_{max} = -16$ and hence $P_{min} = -(Q_{max}) = 16$.

The full working is available in the next frame, should you need to refer to it.

If not, move straight on to Frame 65

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Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	$\boxed{1}$	3	1	0	0	0	54	59
w_2	$\boxed{3}$	1	0	1	0	0	34	$\boxed{39}$
w_4	$\boxed{-1}$	2	0	0	-1	1	12	13
Q	-3	4	0	0	0	M	0	$M+1$
w_1	1	3	1	0	0	0	54	59
w_2	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{34}{3}$	13
w_4	-1	2	0	0	-1	1	12	13
Q	-3	4	0	0	0	M	0	$M+1$
w_1	0	$\frac{8}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{128}{3}$	46
x $\cancel{w_2}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{34}{2}$	13
* → w_4	0	$\boxed{\frac{7}{3}}$	0	$\frac{1}{3}$	-1	1	$\frac{70}{3}$	26
Q	0	5	0	1	0	M	34	$M+40$
w_1	0	$\frac{8}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{128}{3}$	46
x	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{34}{3}$	13
w_4	0	1	0	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	10	$\frac{78}{7}$
Q	0	5	0	1	0	M	34	$M+40$
w_1	0	0	1	$-\frac{5}{7}$	$\frac{8}{7}$	$-\frac{8}{7}$	16	$\frac{114}{7}$
x	1	0	0	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	8	$\frac{65}{7}$
y $\cancel{w_4}$	0	1	0	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	10	$\frac{78}{7}$
Q	0	0	0	$\frac{2}{7}$	$\frac{15}{7}$	$M-\frac{15}{7}$	-16	$M-\frac{110}{7}$

$Q_{max} = -16 \quad \therefore \; P_{min} = 16$ with $x = 8, y = 10$.

65

Example 2

Minimise $\quad P = -2x + 8y$

subject to
$$\begin{aligned} 3x + 4y &\leq 80 \\ -3x + 4y &\geq 8 \\ x + 4y &\geq 40 \quad (x, y \geq 0) \end{aligned}$$

Note that we have two constraints that are 'greater than' inequalities, so, beside the slack variables, we shall need two artificial variables.

The three constraints in their new form therefore become

....

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$$\boxed{\begin{array}{lllll} 3x + 4y + w_1 & & & & = 80 \\ -3x + 4y & - w_2 & & + w_4 & = 8 \\ x + 4y & & - w_3 & + w_5 & = 40 \end{array}}$$

and, in the subsequent manipulation, we must see that w_4 and w_5 disappear from the basic solution before the optimal solution is obtained.

The objective function P is now replaced by $Q\ (= -P)$ and the new form of Q is written as

....

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$$\boxed{Q - 2x + 8y + Mw_4 + Mw_5 = 0}$$

because

$P = -2x + 8y \quad \therefore \quad Q = -P = 2x - 8y$

and with the artificial variables $Q = 2x - 8y \ - Mw_4 - Mw_5$.

$$\therefore \quad Q - 2x + 8y + Mw_4 + Mw_5 = 0$$

In this example, w_1, w_4, w_5 will form the unity matrix, so work through the solution in the usual way. Simplify the initial tableau and then refer to the next frame.

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Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	check
w_1	$\boxed{3}$	4	1	0	0	0	0	80	$\boxed{88}$
w_4	-3	4	0	-1	0	1	0	8	9
w_5	1	4	0	0	-1	0	1	40	45
Q	-2	8	0	0	0	M	M	0	$2M+6$
w_1	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
w_4	-3	4	0	-1	0	1	0	8	9
w_5	1	4	0	0	-1	0	1	40	45
Q	-2	8	0	0	0	M	M	0	$2M+6$
x ~~w_1~~	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
* → w_4	0	8	1	-1	0	1	0	88	97
* → w_5	0	$\frac{8}{3}$	$-\frac{1}{3}$	0	-1	0	1	$\frac{40}{3}$	$\frac{47}{3}$
Q	0	$\frac{32}{3}$	$\frac{2}{3}$	0	0	M	M	$\frac{160}{3}$	$2M+\frac{194}{3}$

At this stage, there is no further negative entry in the index row, but we still must get rid of w_4 and w_5 from the basic variable column. Let us start by dealing with w_5.

We will take the entry $\frac{8}{3}$ at the intersection of the w_5 row and the y column as the next pivot and launch forth on the next stage. Complete the second stage and then again refer to the next frame.

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Here is the working of stage 2.

Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	check
x	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
w_4	0	8	1	-1	0	1	0	88	97
w_5	0	$\boxed{\frac{8}{3}}$	$-\frac{1}{3}$	0	-1	0	1	$\frac{40}{3}$	$\frac{47}{3}$
Q	0	$\frac{32}{3}$	$\frac{2}{3}$	0	0	M	M	$\frac{160}{3}$	$2M+\frac{194}{3}$
x	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
w_4	0	8	1	-1	0	1	0	88	97
w_5	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	$\frac{32}{3}$	$\frac{2}{3}$	0	0	M	M	$\frac{160}{3}$	$2M+\frac{194}{3}$
x	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	20	$\frac{43}{2}$
* → w_4	0	0	2	-1	3	1	-3	48	50
y ~~w_5~~	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	0	2	0	4	M	$M-4$	0	$2M+2$

▶

At this point, w_5 is replaced by y in the basic variable column.

Now we deal with w_4 by taking the entry 2 at the junction of the w_4 row and the w_1 column as the next pivot. That should do the trick, so finish off the solution and check with the next frame.

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$$\boxed{P_{\min} = 48 \quad \text{with} \quad x = 8,\ y = 8}$$

Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	*check*
x	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	20	$\frac{43}{2}$
* $\rightarrow w_4$	0	0	$\boxed{2}$	-1	3	1	-3	48	50
y	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	0	2	0	4	M	$M-4$	0	$2M+2$
x	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	20	$\frac{43}{2}$
w_4	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	24	25
y	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	0	2	0	4	M	$M-4$	0	$2M+2$
x	1	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	8	9
w_1 $\cancel{w_4}$	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	24	25
y	0	1	0	$-\frac{1}{16}$	$-\frac{3}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	8	9
Q	0	0	0	1	1	$M-1$	$M-1$	-48	$2M-48$

w_4 in the basic variable column is now replaced by w_1, so the conditions are satisfied at last. From the final tableau, we have

$$Q_{\max} = -48 \qquad \text{But } P_{\min} = -(Q_{\max}) = 48$$

$$\therefore\ P_{\min} = 48 \quad \text{with} \quad x = 8,\ y = 8.$$

By this means, then, we can solve minimisation problems by the simplex method and so widen the scope of this valuable technique.

Applications

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So far we have seen how to solve a typical linear programming problem by the simplex method, when the data are presented as a linear objective function and a number of linear constraints in the form of equations or inequalities. A practical problem, however, must first be interpreted into algebraic form and we conclude the Programme with a brief reference to this initial requirement. Let us consider the following example.

▶

Example 1

A firm manufactures two types of couplings, A and B, each of which requires processing time on lathes, grinders and polishers. The machine times needed for each type of coupling are given in the table.

Coupling type	*Time required (hours)*		
	Lathe	*Grinder*	*Polisher*
A	2	8	5
B	5	5	2

The total machine time available is 250 hours on lathes, 310 hours on grinders and 160 hours on polishers. The net profit per coupling of type A is £9 and of type B £10.

Determine

(a) the number of each type to be produced to maximise profit

(b) the maximum profit.

If we let $x =$ the number of type A units to be produced

$y =$ the number of type B units to be produced

the objective function to be maximised can be expressed as ....

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$$\boxed{P = 9x + 10y}$$

Now we have to sort out the constraints from the given data.

Total time available on lathes = 250 hours

$\therefore\ 2x + 5y \leq 250$ (lathes)

Similar statements for the grinders and polishers are ....

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$$\boxed{\begin{array}{l} 8x + 5y \leq 310 \ \text{(grinders)} \\ 5x + 2y \leq 160 \ \text{(polishers)} \end{array}}$$

The problem now can be expressed as

$$\begin{array}{ll} \text{Maximise} & P = 9x + 10y \\ \text{subject to} & 2x + 5y \leq 250 \\ & 8x + 5y \leq 310 \\ & 5x + 2y \leq 160 \quad (x, y \geq 0) \end{array}$$

Then we go through the usual process. Inserting slack variables to convert the inequalities into equations, we have ....

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$$\begin{array}{|l|}\hline \begin{aligned} 2x + 5y + w_1 \qquad\qquad &= 250 \\ 8x + 5y \qquad + w_2 \qquad &= 310 \\ 5x + 2y \qquad\qquad + w_3 &= 160 \\ P - 9x - 10y \qquad\qquad &= 0 \end{aligned} \\ \hline \end{array}$$

and the solution then develops in the usual way. Work through it carefully – it is all good practice – and see if you agree with the result given in the next frame.

The result is ....

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$$\boxed{P_{max} = 550 \quad \text{with} \quad x = 10, y = 46}$$

The maximum profit of £550 occurs with a manufacturing schedule of

10 couplings of type A

and 46 couplings of type B.

Now for another, so move on.

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Example 2

A firm produces three types of pumps, A, B, C, each of which requires the four processes of turning, drilling, assembling and testing.

Pump type	*Process time (hours) per pump*				*Profit per pump* £
	Turning	*Drilling*	*Assembling*	*Testing*	
A	2	1	3	4	84
B	1	1	4	3	72
C	2	1	2	2	52
Total available time (h/week)	98	60	145	160	

From the information given in the table, determine

(a) the weekly output of each type of pump to maximise profit

(b) the maximum profit.

So, if we let $x =$ the number of pumps, type A

$y =$ the number of pumps, type B

$z =$ the number of pumps, type C

we can interpret the problem into its algebraic form, which is

....

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$$\begin{array}{ll}\text{Maximise} & P = 84x + 72y + 52z \\ \text{subject to} & 2x + y + 2z \leq 98 \\ & x + y + z \leq 60 \\ & 3x + 4y + 2z \leq 145 \\ & 4x + 3y + 2z \leq 160 \qquad (x, y, z \geq 0)\end{array}$$

Inserting the slack variables and expressing the problem as equations, we have

....

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$$\begin{array}{rl} 2x + y + 2z + w_1 & = 98 \\ x + y + z + w_2 & = 60 \\ 3x + 4y + 2z + w_3 & = 145 \\ 4x + 3y + 2z + w_4 & = 160 \\ P - 84x - 72y - 52z & = 0 \end{array}$$

Now you can proceed to set up the simplex tableau and solve the problem on your own in the usual manner. It is very similar to the other examples you have worked earlier in the Programme.

The result you no doubt get is

....

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$$\boxed{P_{max} = 3652 \quad \text{with} \quad x = 23, y = 8, z = 22}$$

i.e. by producing 23 pumps, type A
8 pumps, type B
22 pumps, type C

the maximum profit of £ 3652 is attained.

Care with the calculations and constant use of the check column provide the key to avoiding errors in the working.

That completes the Programme. Check down the **Revision summary** that comes next, in conjunction with the **Can You?** checklist, before working through the **Test exercise** that follows thereafter. As usual, a set of **Further problems** provides further necessary practice in these useful techniques.

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Revision summary 23

1 *Optimization* – determination of an optimal value (maximum or minimum) of an objective function subject to a set of constraints.

2 *Linear programming (linear optimization)* – optimization where the objective function is a linear function and the constraints are linear equations or linear inequalities.

3 *Inequalities* – multiplying or dividing both sides by a negative factor $(-k)$ reverses the inequality, i.e. $\geq$ becomes $\leq$ and $\leq$ becomes $\geq$.

4 *Problem variables* $(x, y, z$, etc.) are always non-negative.

5 *Feasible solution* – a set of variables that satisfies all the given constraints.

6 *Optimal solution* – a feasible solution for which the objective function becomes a maximum (or minimum) within the constraints.

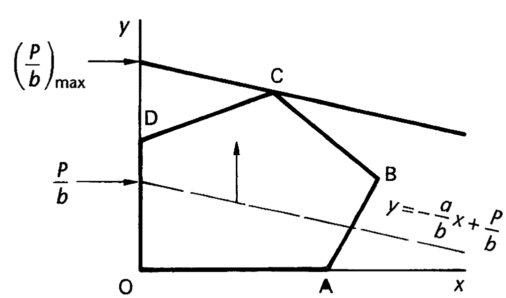
7 *Basic feasible solution* – a feasible solution for which at least $(n - m)$ of the total variables are zero, where
$n =$ total number of variables in the constraints
$m =$ number of constraints.

8 *Basis* – collection of the m variables which are not put equal to zero.

9 *Basic solution* – solution obtained by equating $(n - m)$ variables to zero and solving for the remaining m variables.

10 *Graphical solution*

(a) Constraints – graphs of constraints form the feasible polygon or feasible domain.



Feasible point or feasible solution – coordinates of all points within the feasible polygon or on its boundary (OABCD).

(b) Objective function $P = ax + by$ $\therefore$ $y = -\frac{a}{b}x + \frac{P}{b}$ represented by a set of parallel lines, slope $-\frac{a}{b}$, intercept $\frac{P}{b}$. Line through the extreme point C gives P_{max}, the optimal value of P.

▶

11 *Slack variable* – non-negative variable added to, or subtracted from, a linear inequality to form a linear equation.

12 *Simplex method of solution* – computation.

Refer back to Frame 34.

Where necessary, we multiply an inequality by (–1), with consequent reversal of inequality sign, to ensure that the right-hand side constant term $b_i \geq 0$.

13 *Artificial variable* – to convert a 'greater than' inequality to an equation, the slack variable required must be subtracted. To complete the unity matrix in the tableau, a further artificial variable w_i is included to allow the simplex procedure to continue. Such artificial variables must be eliminated before the optimal solution is finally attained.

The objective function $P = ax + by$ becomes $P = ax + by - Mw_i$.

14 *Minimisation* – If P is the objective function to be minimised

(a) write $Q = -P$

(b) maximise Q by the usual simplex method

(c) then $Q_{max} = (-P)_{max} = -(P_{min})$

i.e. $P_{min} = -(Q_{max})$.

Can You?

81 Checklist 23

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Describe an optimization problem in terms of the objective function and a set of constraints? 1 and 2

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Algebraically manipulate and graphically describe inequalities? 3 to 6

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve a linear programming problem in two real variables? 6 to 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the simplex method to describe a linear programming problem in two real variables as a problem in two real variables with two slack variables? 14

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Set up the simplex tableau and compute the simplex? 15 to 32

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the simplex method to solve a linear programming problem in three real variables with three slack variables? 33 to 40

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Introduce artificial variables into the solution method as and when the need arises? 41 to 60

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve minimisation problems using the simplex method? 61 to 70

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Construct the algebraic form of the objective function and the constraints for a problem stated in words? 71 to 79

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 23

82

1 Using a *graphical method*, maximise $P = x + 2y$ subject to the constraints

$$\begin{aligned} -3x + 4y &\leq 8 \\ x + 4y &\leq 16 \\ 3x + 2y &\leq 18 \\ x, y &\geq 0. \end{aligned}$$

Note: Use the *simplex method* to solve Exercises **2** to **6**. In each case, all variables are non-negative.

2 Maximise $P = -3x + 4y$
subject to
$$\begin{aligned} 3x - 2y &\leq 15 \\ x + y &\leq 10 \\ -x + 4y &\leq 15 \\ -2x + y &\leq 2. \end{aligned}$$

3 Maximise $P = 8x + 12y + 10z$
subject to
$$\begin{aligned} 4x + 3y + 2z &\leq 64 \\ 2x + y + 4z &\leq 48 \\ x + 2y + z &\leq 24. \end{aligned}$$

4 Maximise $P = 44x + 20y$
subject to
$$\begin{aligned} 12x + 6y &\leq 84 \\ 3x + 2y &\geq 24. \end{aligned}$$

5 Minimise $P = 3y - 4x$
subject to
$$\begin{aligned} x + 4y &\leq 60 \\ 2x + y &\leq 22 \\ -x + y &\geq 7. \end{aligned}$$

▶

6 A firm makes two types of containers, A and B, each of which requires cutting, assembly and finishing. The maximum available machine capacity in hours per week for each process is: cutting 50, assembly 84, finishing 72.

The process times for one unit of each type are as follows:

Process	*Time in hours* A	*Time in hours* B
Cutting	2	5
Assembly	4	8
Finishing	4	5

If the profit margin is £600 per unit A and £1000 per unit B, determine

(a) the optimum weekly output of containers

(b) the maximum profit.

Further problems 23

83

All variables in the following problems are non-negative.

Graphical Solution

1 Maximise $P = -x + 8y$

subject to
$$-3x + 4y \leq 10$$
$$-x + 4y \leq 14$$
$$3x + 2y \leq 21$$
$$3x + y \leq 18.$$

2 Maximise $P = -4x + 8y$

subject to
$$x + 3y \leq 57$$
$$7x + 4y \leq 110$$
$$-x + 5y \leq 40.$$

3 Maximise $P = 5x + 4y$

subject to
$$x - 2y \leq 2$$
$$3x - 4y \leq 8$$
$$5x + 6y \leq 45$$
$$x + 3y \leq 18.$$

Simplex Solution

4 Maximise $P = 2x + y$
subject to
$$\begin{aligned} x + 4y &\le 24 \\ x + y &\le 9 \\ x - y &\le 3 \\ x - 2y &\le 2. \end{aligned}$$

5 Maximise $P = -3x + 4y$
subject to
$$\begin{aligned} 3x - 4y &\le 12 \\ 5x + 4y &\le 36 \\ -x + 3y &\le 8 \\ -3x + y &\le 0. \end{aligned}$$

6 Maximise $P = x + 2y$
subject to
$$\begin{aligned} -2x + y &\le 1 \\ -x + y &\le 2 \\ x + y &\le 6 \\ 2x - 3y &\le 2. \end{aligned}$$

7 Maximise $P = 4y - 3x$
subject to
$$\begin{aligned} x - 2y &\le 0 \\ x - y &\le 2 \\ x + 2y &\le 14 \\ -x + 2y &\le 6 \\ -3x + 2y &\le 2. \end{aligned}$$

8 Maximise $P = 3x + 4y + 5z$
subject to
$$\begin{aligned} 5x + 4y + 8z &\le 40 \\ 3x + 2y + 12z &\le 30 \\ y &\le 8. \end{aligned}$$

9 Maximise $P = 3x + 4y + 3z$
subject to
$$\begin{aligned} 2x + 3y + 4z &\le 58 \\ 4x + 2y + 3z &\le 51 \\ 3x + 4y + 2z &\le 62. \end{aligned}$$

10 Maximise $P = 4x + 3y + 3z$
subject to
$$\begin{aligned} 4x + y + 2z &\le 40 \\ x + 4y + z &\le 50 \\ 2x + 3y + 4z &\le 60. \end{aligned}$$

11 Maximise $P = 5{\cdot}3x + 3{\cdot}6y + 2{\cdot}0z$
subject to
$$\begin{aligned} 2{\cdot}1x + 4{\cdot}3y + 1{\cdot}5z &\le 70 \\ 3{\cdot}2x + 1{\cdot}4y + 2{\cdot}2z &\le 60 \\ 1{\cdot}6x + 6{\cdot}2y + 3{\cdot}1z &\le 100. \end{aligned}$$

Artificial Variables

12 Maximise $P = 8x + 5y$
subject to
$$\begin{aligned} 2x + y &\le 80 \\ x + 3y &\le 90 \\ x + y &\ge 30. \end{aligned}$$

13 Maximise $P = 12x + 8y$
subject to
$$\begin{aligned} x + 2y &\le 20 \\ 4x - y &\le 8 \\ -x + y &\ge 1. \end{aligned}$$

14 Maximise $P = 3x + 4y$
subject to
$$\begin{aligned} x + 4y &\le 76 \\ -5x + 8y &\ge 40 \\ -x + 4y &\ge 32. \end{aligned}$$

15 Minimise $P = 4x + 5y$
subject to
$$\begin{aligned} x + 2y &\le 63 \\ 3x + y &\le 70 \\ 2x + y &\ge 42 \\ x + 4y &\ge 84. \end{aligned}$$

16 Maximise $P = 65x - 23y$
subject to
$$\begin{aligned} 5x - y &\le 30 \\ 10x + 4y &\ge 150. \end{aligned}$$

17 Maximise $P = 24x - 8y$
subject to
$$\begin{aligned} x + 3y &\le 360 \\ 2x + y &\le 850 \\ -5x + 25y &\ge 320. \end{aligned}$$

18 Maximise $P = 4x + 2y$
subject to
$$\begin{aligned} x + 2y &\le 60 \\ 3x + 2y &\le 80 \\ -3x + 10y &\ge 40. \end{aligned}$$

19 Maximise $P = 18x + 40y + 24z$
subject to
$$\begin{aligned} 5x + 2y + 4z &\le 63 \\ 2x + 4y + 2z &\le 42 \\ 2x + 3y + z &\ge 35. \end{aligned}$$

20 Maximise $P = 60x + 45y + 25z$
subject to
$$\begin{aligned} 4x + 8y + 2z &\le 160 \\ 6x + 3y + 4z &\le 168 \\ 4x + 3y + 3z &\ge 128. \end{aligned}$$

21 Maximise $P = 12x + 8y - 10z$
subject to
$$\begin{aligned} 4x + 2y - 3z &\le 210 \\ 6x + 8y + z &\le 630 \\ 2x - y + 4z &\ge 210 \\ x + y + z &\le 180. \end{aligned}$$

Minimisation

22 Minimise $P = -4x + 3y$
subject to
$$\begin{aligned} x + 4y &\le 20 \\ 2x + y &\le 12 \\ x - y &\le 3. \end{aligned}$$

23 Minimise $P = -5x + 8y$
subject to
$$\begin{aligned} x + 2y &\le 40 \\ 3x + 2y &\le 48 \\ -x + 4y &\ge 40. \end{aligned}$$

24 Minimise $P = -4x + 8y$
subject to
$$\begin{aligned} -5x + 4y &\le 32 \\ 7x + 4y &\le 80 \\ -x + 8y &\ge 40. \end{aligned}$$

25 Minimise $P = 2x + 8y$
subject to
$$\begin{aligned} -x + 2y &\le 24 \\ 7x + 6y &\le 132 \\ -x + 2y &\ge 4 \\ x + 2y &\ge 12. \end{aligned}$$

26 Minimise $P = 4x - 8y + 5z$
subject to
$$\begin{aligned} 2x + 3y + z &\le 70 \\ x + 2y + 2z &\le 60 \\ 3x + 4y + z &\le 84 \\ x + y + z &\ge 33. \end{aligned}$$

27 Minimise $P = 6x - 5y - 3z$
subject to
$$\begin{aligned} 5x + 8y + 4z &\le 220 \\ 2x + y + 6z &\le 154 \\ 4x + 2y + z &\ge 77 \\ x + y + 2z &\ge 55. \end{aligned}$$

▶

Applications

28 A firm manufacturing two types of switching module, A and B, is under contract to produce a daily output of at least 35 modules in all. Assembly and testing times for each type of module are as follows:

Module type	*Processing time (hours)*	
	Assembly	*Testing*
A	1·0	2·0
B	2·0	1·0

Available staff resources provide a daily maximum of 80 hours for assembly and 55 hours for testing.
The profit on the sale of each A-module is £40 and of each B-module £50. Determine

(a) the daily production schedule for maximum profit.

(b) the maximum daily profit.

29 Three different types of coupling units are produced by a firm. The times required for machining, polishing and assembling a unit of each type are included in the information given in the following table.

Type of unit	*Process time (hours) per unit*			*Profit (£) per unit*
	Machining	*Polishing*	*Assembling*	
A	4	1	2	110
B	2	3	1	100
C	3	2	4	120
Available time (h/week)	320	250	280	

The firm is required to supply a total of at least 100 units of mixed types each week. Determine

(a) the weekly output of each type to maximise profit

(b) the maximum weekly profit.

▶

30 A firm makes three types of wooden cabinets, A, B, C, with profit margins of £35, £30, £24 per unit respectively.

Process	*Time in hours per cabinet*		
	A	B	C
Preparation	2	5	4
Assembly	2	3	2
Finishing	5	4	3

The manufacturer has 25 men available for preparation, 20 men for assembly and 30 men for polishing, and all staff work a 40 hour week. To remain competitive, at least 300 cabinets in all must be produced each week. Determine

(a) the number of each model to be manufactured each week in order to maximise the profit

(b) the maximum weekly profit.

Appendix

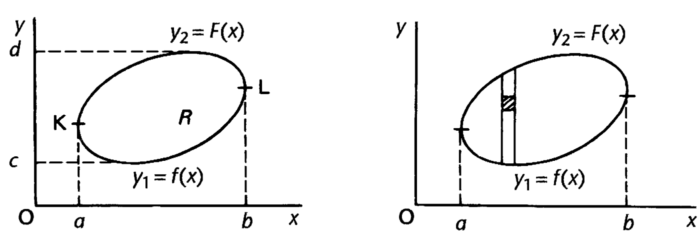
1 Green's theorem

If P and Q are two functions in x and y, finite and continuous inside a region R and on its boundary c in the x–y plane, with continuous first partial derivatives, then Green's theorem states that

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \mathrm{d}x\,\mathrm{d}y = -\oint_c \{P\mathrm{d}x + Q\mathrm{d}y\}$$

Proof of Green's theorem

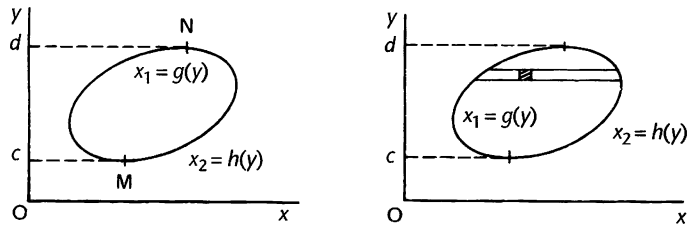
Let the lower boundary of the region be the curve $y_1 = f(x)$ and the upper boundary the curve $y_2 = F(x)$.



Using vertical strips, we then have

$$\begin{aligned}
\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \int_{y_1}^{y_2} \frac{\partial P}{\partial y}\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \Big[P\Big]_{y_1=f(x)}^{y_2=F(x)} \mathrm{d}x \\
&= \int_a^b \{P(x,y_2) - P(x,y_1)\}\,\mathrm{d}x \\
&= -\int_a^b P(x,y_1)\,\mathrm{d}x - \int_b^a P(x,y_2)\,\mathrm{d}x \\
&= -\left\{\int_a^b P(x,y_1)\,\mathrm{d}x + \int_b^a P(x,y_2)\,\mathrm{d}x\right\} \\
&= -\oint P(x,y)\,\mathrm{d}x \qquad (1)
\end{aligned}$$

Similarly, using horizontal strips, we have



$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d \int_{x_1}^{x_2} \frac{\partial Q}{\partial y}\,dx\,dy$$

$$= \int_c^d \Big[Q \Big]_{x_1=g(y)}^{x_2=h(y)} dy$$

where $x_1 = g(y)$ and $x_2 = h(y)$ are the left-hand and right-hand portions of the boundary curve c.

$$\therefore \quad \iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d Q(x_2, y)\,dy - \int_c^d Q(x_1, y)\,dy$$

$$= \int_c^d Q(x_2, y)\,dy + \int_d^c Q(x_1, y)\,dy$$

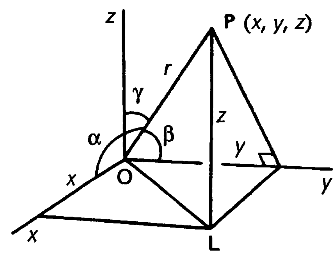
$$= \oint_c Q(x, y)\,dy \qquad (2)$$

$$\therefore \quad \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_c P(x, y)\,dx - \oint_c Q(x, y)\,dy$$

$$= -\oint_c \{P\,dx - Q\,dy\}$$

2 Proof that $\sec\gamma = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$

Let α, β, γ be the angles that OP makes with the x, y and z axes respectively.



Then $x = r\cos\alpha$; $y = r\cos\beta$; $z = r\cos\gamma$

Also $x^2 + y^2 + z^2 = r^2$

If $r = 1$ unit, then $x^2 + y^2 + z^2 = 1 \quad \therefore\ z^2 = 1 - x^2 - y^2$

$$\therefore\ z = (1 - x^2 - y^2)^{1/2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y)$$

$$= \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

$$\therefore\ 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2}$$

$$= \frac{1 - x^2 - y^2 + x^2 + y^2}{1 - x^2 - y^2}$$

$$= \frac{1}{1 - x^2 - y^2} = \frac{1}{z^2}$$

But, with $r = 1$, $z = \cos\gamma \qquad \therefore\ \frac{1}{z^2} = \sec^2\gamma$

$$\therefore\ \sec\gamma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

3 Vector triple products

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$

(b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\,\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\,\mathbf{A}$

Let $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$;

$\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$

Then $\mathbf{B} \times \mathbf{C} = (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \times (c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}$$

Then
$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} & \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} & \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \end{vmatrix}$$

$$= \mathbf{i}\left\{ a_y\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} - a_z\begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} \right\} - \mathbf{j}\left\{ a_x\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} - a_z\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} \right\}$$

$$+ \mathbf{k}\left\{ a_x\begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} - a_y\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} \right\}$$

$$= \mathbf{i}\,\{a_y(b_xc_y - b_yc_x) - a_z(b_zc_x - b_xc_z)\}$$
$$+ \mathbf{j}\,\{a_z(b_yc_z - c_yb_z) - a_x(b_xc_y - b_yc_x)\}$$
$$+ \mathbf{k}\,\{a_x(b_zc_x - b_xc_z) - a_y(b_yc_z - b_zc_y)\}$$

$$= \mathbf{i}\,\{b_xa_xc_x + b_xa_yc_y + b_xa_zc_z - c_xa_xb_x - c_xa_yb_y - c_xa_zb_z\}$$
$$+ \mathbf{j}\,\{b_ya_xc_x + b_ya_yc_y + b_ya_zc_z - c_ya_xb_x - c_ya_yb_y - c_ya_zb_z\}$$
$$+ \mathbf{k}\,\{b_za_xc_x + b_za_yc_y + b_za_zc_z - c_za_xb_x - c_za_yb_y - c_za_zb_z\}$$

$$= \mathbf{i}\,\{b_x(a_xc_x + a_yc_y + a_zc_z) - c_x(a_xb_x + a_yb_y + a_zb_z)\}$$
$$+ \mathbf{j}\,\{b_y(a_xc_x + a_yc_y + a_zc_z) - c_y(a_xb_x + a_yb_y + a_zb_z)\}$$
$$+ \mathbf{k}\,\{b_z(a_xc_x + a_yc_y + a_zc_z) - c_z(a_xb_x + a_yb_y + a_zb_z)\}$$

Now $\quad \mathbf{A}\cdot\mathbf{C} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k})$

$\qquad\qquad = a_xc_x + a_yc_y + a_zc_z$

and similarly $\quad \mathbf{A}\cdot\mathbf{B} = a_xb_x + a_yb_y + a_zb_z$

$$\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{i}\,\{b_x(\mathbf{A}\cdot\mathbf{C}) - c_x(\mathbf{A}\cdot\mathbf{B})\} + \mathbf{j}\,\{b_y(\mathbf{A}\cdot\mathbf{C}) - c_y(\mathbf{A}\cdot\mathbf{B})\} + \mathbf{k}\,\{b_z(\mathbf{A}\cdot\mathbf{C}) - c_z(\mathbf{A}\cdot\mathbf{B})\}.$$

$$\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\{\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z\} - (\mathbf{A}\cdot\mathbf{B})\,\{\mathbf{i}c_x + \mathbf{j}c_y + \mathbf{k}c_z\}$$

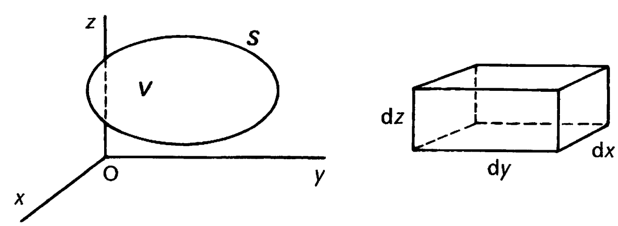
$$\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$$

In the same way, it can be established that

$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\,\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\,\mathbf{A}$

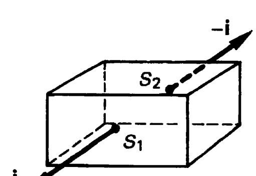
4 Divergence theorem (Gauss' theorem)

To prove that $\int_V \operatorname{div}\mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ for the region V bounded by the surface S.



Consider an element of volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and let the components of $\mathbf{F}$ in the x, y and z directions be denoted by $F_x\mathbf{i}$, $F_y\mathbf{j}$ and $F_z\mathbf{k}$ respectively at any point P. We then determine $\int \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ over the element $\mathrm{d}V$ and finally sum the results for all such elements throughout the region.

(a) S_1: $\quad \mathrm{d}S_1 = \mathrm{d}y\,\mathrm{d}z; \quad \mathbf{n} = \mathbf{i}$



$$(\mathbf{F}\cdot\mathrm{d}\mathbf{S})_1 = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{n})\,\mathrm{d}S_1$$
$$= (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{i})\,\mathrm{d}S_1$$
$$= F_x\,\mathrm{d}S_1$$

(b) S_2 : $\quad dS_2 = dy\,dz; \quad \mathbf{n} = -\mathbf{i}$

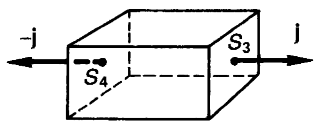
$$\therefore \quad (\mathbf{F}\cdot d\mathbf{S})_2 = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(-\mathbf{i})\,dS_2 = -F_x\,dS_2$$

Combining these two results, we have

$$(\mathbf{F}\cdot d\mathbf{S})_1 + (\mathbf{F}\cdot d\mathbf{S})_2 = (F_x\,dS)_1 - (F_x\,dS)_2 = \frac{\partial}{\partial x}(F_x\,dS)\,dx$$

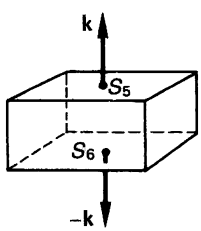
$$\therefore \quad \int_{S_1+S_2} \mathbf{F}\cdot d\mathbf{S} = \frac{\partial F_x}{\partial x}\,dS\,dx = \left(\frac{\partial F_x}{\partial x}\right)dx\,dy\,dz \qquad (1)$$

Similarly, for S_3 and S_4 we have



$$\int_{S_3+S_4} \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_y}{\partial y}\right)dx\,dy\,dz \qquad (2)$$

and for S_5 and S_6



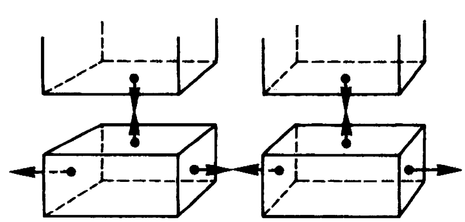
$$\int_{S_5+S_6} \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_z}{\partial z}\right)dx\,dy\,dz \qquad (3)$$

These three results together cover the total surface of the element dV.

$$\int_{S_1\ldots S_6} \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dx\,dy\,dz = \operatorname{div}\mathbf{F}\,dV$$

Finally, summing the results for all such elements throughout the region with $dV \to 0$ and $d\mathbf{S} \to 0$, we obtain

$$\int_V \operatorname{div}\mathbf{F}\,dV = \sum \int \mathbf{F}\cdot d\mathbf{S} \quad \text{with } d\mathbf{S} \to 0$$

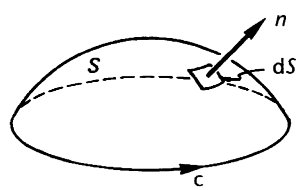


On the common boundaries between adjacent elements, the values of $\int \mathbf{F}\cdot d\mathbf{S}$ cancel out. On the boundary surface, however, there are no such adjacent faces and the integral $\oint_S \mathbf{F}\cdot d\mathbf{S}$ remains.

$$\therefore \int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

5 Stokes' theorem

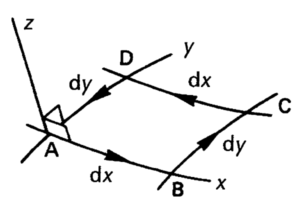
If $\mathbf{F}$ is a single-valued vector field, continuous and differentiable over an open surface S and on the boundary c of the surface, then



$$\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Proof of Stokes' theorem

Consider the surface S divided into small rectangular elements and let ABCD be one such element. If axes of reference x and y be arranged to coincide with AB and AD respectively as shown, a third axis z will then be normal to the surface at A.



If AB $= dx$, then $d\mathbf{x} = \mathbf{i}\,dx$ and

if AD $= dy$, then $d\mathbf{y} = \mathbf{j}\,dy$.

Let $\mathbf{F}_a$ denote the vector field at A; $\mathbf{F}_b$ that at B; $\mathbf{F}_c$ that at C; and $\mathbf{F}_d$ that at D. Now consider each side in turn.

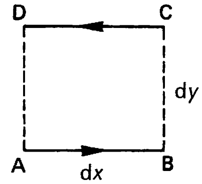
AB: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_a\cdot d\mathbf{x} = \{F_{ax}\mathbf{i} + F_{ay}\mathbf{j} + F_{az}\mathbf{k}\}\cdot\{\mathbf{i}\,dx\} = F_{ax}\,dx$

BC: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_b\cdot d\mathbf{y} = \{F_{bx}\mathbf{i} + F_{by}\mathbf{j} + F_{bz}\mathbf{k}\}\cdot\{\mathbf{j}\,dy\} = F_{by}\,dy$

CD: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_c\cdot d\mathbf{x} = \{F_{cx}\mathbf{i} + F_{cy}\mathbf{j} + F_{cz}\mathbf{k}\}\cdot\{-\mathbf{i}\,dx\} = -F_{cx}\,dx$

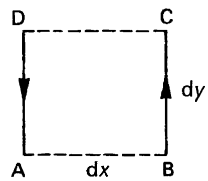
DA: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_d\cdot d\mathbf{y} = \{F_{dx}\mathbf{i} + F_{dy}\mathbf{j} + F_{dz}\mathbf{k}\}\cdot\{-\mathbf{j}\,dy\} = -F_{dy}\,dy$

(a) AB + CD:



$$\int \mathbf{F}\cdot \mathrm{d}\mathbf{r} = F_{ax}\,\mathrm{d}x - F_{cx}\,\mathrm{d}x$$
$$= -(F_{cx} - F_{ax})\,\mathrm{d}x$$
$$= -\,\delta F_x\,\mathrm{d}x$$
$$= -\frac{\partial F_x}{\partial y}\,\mathrm{d}y\,\mathrm{d}x$$
$$\therefore \quad \int_{(AB+CD)} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\frac{\partial F_x}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \qquad (1)$$

(b) BC + DA:



$$\int \mathbf{F}\cdot \mathrm{d}\mathbf{r} = F_{by}\,\mathrm{d}y - F_{dy}\,\mathrm{d}y$$
$$= (F_{by} - F_{dy})\,\mathrm{d}y$$
$$= \delta F_y\,\mathrm{d}y$$
$$= \frac{\partial F_y}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$$
$$\therefore \quad \int_{(BC+DA)} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \frac{\partial F_y}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \qquad (2)$$

Adding these two results together for the complete rectangle, we have

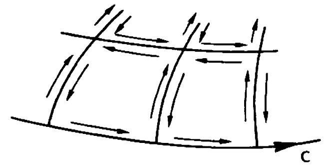
$$\int_{(ABCD)} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \left\{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right\}\mathrm{d}x\,\mathrm{d}y \qquad (3)$$

$$\text{Now curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$
$$= \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$
$$\therefore \quad \left\{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right\} = (\text{curl } \mathbf{F})\cdot(\mathbf{k}) \qquad (4)$$

From (3)
$$\int_{ABCD} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \text{curl } \mathbf{F}\cdot\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y = \text{curl } \mathbf{F}\cdot \mathrm{d}\mathbf{S}$$

Summing for all such elements over the surface

$$\int_S \text{curl } \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \mathop{Lim}_{\mathrm{d}\mathbf{r}\to 0} \sum \left\{ \int_{ABCD} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \right\} \qquad (5)$$



$\int \mathbf{F} \cdot d\mathbf{r}$ on boundary lines between adjacent rectangular elements will cancel out, except on the boundary curve c of the surface S. The integral then becomes $\oint_c \mathbf{F} \cdot d\mathbf{r}$.

$$\therefore \quad \int_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$$

Answers

Test exercise 1 (page 42)

1 $x = -1 - j\sqrt{3}$; $x^2 + 2x + 4 = 0$ **2** $x = -4, 6, 3/2$ **3** $x = -1\cdot\dot{6} = -5/3$
4 $x \approx 1\cdot710$ **5** $x \approx 0\cdot454304$ **6** $x \approx 1\cdot317672$ **7** (a) 39·375
(b) 103·392 (c) 481·528 **8** −12·8

Further problems 1 (page 43)

1 $\dfrac{-1 + j\sqrt{3}}{2}$, $\dfrac{-1 - j}{\sqrt{2}}$, $x^4 + (1 + \sqrt{2})x^3 + (2 + \sqrt{2})x^2 + (1 + \sqrt{2})x + 1 = 0$
2 $x = 1, 6, -2$ **3** $p = -5, q = -1$ **4** $p = 4, q = 9$ **5** $x = 2, 3, -3$
6 $x = 1, -3, 9$ **7** $y^3 - 5y^2 + 17y - 13 = 0$ **8** $y^3 - 13y^2 + 52y - 60 = 0$
9 $x = \frac{1}{2}, \frac{3}{2}, -1$ **10** $x = -2, 4, 8$ **11** $2y^3 - 15y^2 + 25y = 0$ **13** 0·8934
14 $x = 2\cdot732, -0\cdot732, -2\cdot000$ **15** $y^3 - 3y + 2 = 0$; $x = -4, -1, -1$
16 $x = 1\cdot646$ **17** (a) −0·6736 (b) 0·3717
18 (a) −2·3301, 0·2016, 2·1284 (b) $1, -0\cdot50 \pm j1\cdot66$
(c) −2·115, 0·254, 1·861 **19** (a) $-4\cdot104, -0\cdot9481 \pm j0\cdot5652$
(b) 0·5, −1·5, −1·5 (c) $0\cdot25, 1 \pm j3$ **20** (a) −2·456 (b) 1·765
(c) 0·739 (d) 1·812 (e) 1·8175 (f) 0·5170 (g) 0·8449 (h) 0·8806
21 (a) 32·872 (b) 204·328 (c) 381·429 **22** (a) −1·375 and 81·104
(b) 136·971 and −363·429 **23** (a) −6·048 (b) 461·496
24 (a) 133 and −9·048 (b) 0·136 and −65·433 (c) −199·112 and −867
25 0·02768 **26** −1·0670 **27** (a) −2·54846 (b) −2·41734 (c) −1·87134

Test exercise 2 (page 90)

1 (a) $\dfrac{32 - 2s}{s^2 - 16}$ (b) $\dfrac{s + 4}{s^2 + 16}$ (c) $\dfrac{1}{s^4}\{4s^3 - s^2 + 4s + 6\}$ (d) $\dfrac{s + 2}{s^2 + 4s + 29}$
(e) $\dfrac{6s}{(s^2 + 9)^2}$ (f) $\ln\left\{\dfrac{s + 2}{s + 1}\right\}$ **2** (a) $2e^{3t} - e^{4t}$
(b) $2\cos\sqrt{2}t + \dfrac{5}{\sqrt{2}}\sin\sqrt{2}t - e^t$ (c) $e^t(3t + 2) - e^{3t}$
(d) $\frac{1}{8}\{e^t(17\cos 2t + 9\sin 2t) - e^{3t}\}$ **3** (a) $x = e^{-2t} + e^{-3t}$
(b) $x = \frac{1}{12}\{13e^{2t} - \cos 2t - \sin 2t\}$ (c) $x = \frac{1}{6} - \frac{5}{3}e^{3t} + \frac{5}{2}e^{4t}$
(d) $x = e^t\left(1 - t + \dfrac{t^3}{6}\right)$
4 $x = \frac{1}{2}\{9\cos t - 7\sin t - e^{-3t}\}$ $y = 3\sin t - 2\cos t + e^{-2t}$

Further problems 2 (page 91)

1 (a) $\dfrac{s - 4}{s^2 - 8s + 20}$ (b) $\dfrac{4s}{(s^2 + 4)^2}$ (c) $\dfrac{6}{s^4} + \dfrac{8}{s^3} + \dfrac{5}{s}$ (d) $\dfrac{4s^2 - 24s + 38}{(s - 3)^3}$
(e) $\dfrac{2s^3 - 6s}{(s^2 + 1)^3}$ (f) $\ln\sqrt{\dfrac{s + 2}{s - 2}}$ **2** (a) $e^{2t} + e^{4t}$ (b) $3e^{4t} + 2$
(c) $e^{2t}\left\{\dfrac{3t^2}{2} + 2t + 1\right\}$ (d) $e^{-t}\{2\cos t - 5\sin t\} - 2e^{2t}$

(e) $\frac{1}{3}(\cos t - \cos 2t)$ (f) $e^{-2t}\{\cos 4t - \frac{7}{4}\sin 4t\}$ **3** $x = 4e^{4t} - 2$

4 $x = \frac{35}{78}e^{4t/3} - \frac{3}{26}\{\cos 2t + \frac{2}{3}\sin 2t\}$ **5** $x = e^t(2t+1) + 2t + 4 + \cos t$

6 $x = \frac{3}{2}e^{4t} - e^{3t} - \frac{1}{2}e^{2t}$ **7** $x = \frac{4}{5}\cos 3t + \sin 3t + \frac{1}{5}\cos 2t$

8 $x = \frac{1}{5}\{e^{2t} - e^t(\cos 2t - 2\sin 2t)\}$ **9** $x = \frac{1}{8}\{2t^2 - 4t + 3 + e^{-2t}(4t^2 + 6t + 1)\}$

10 $x = \frac{2}{5}\{2(e^{-4t} - 1)\cos 4t + (e^{-4t} + 1)\sin 4t\}$ **11** $x = (2t+1)\cos 5t + t\sin 5t$

12 $x = \frac{1}{13}\{2e^{2t} + 3e^{-2t} - 5(\cos 3t - \sin 3t)\}$

$y = \frac{1}{13}\{5(\cos 3t + \sin 3t) - 3e^{2t} - 2e^{-2t}\}$

13 $x = \frac{1}{6}\{7e^{-6t} + 5\}$ $y = \frac{1}{3}\{7e^{-6t} + 5\}$ **14** $x = 10e^{-4t} + 2$ $y = 5e^{-4t} + 3$

15 $x = e^{-2t} - e^t + 2t$ $y = 3e^t + \frac{1}{2}e^{-2t} + t - \frac{7}{2}$ **16** $x = 5e^t + 3e^{-t}$ $y = 4e^t - e^{-t}$

17 $x = 4\cos t - 2\sin t - \frac{1}{3}\{8e^{-t} + e^{2t}\}$ $y = 6\cos t + 2\sin t - \frac{4}{3}\{2e^{-t} + e^{2t}\}$

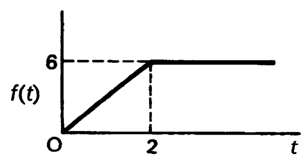
18 $x = \frac{5}{3}\{\cos 2t + \sin 2t - \cosh\sqrt{2}t - \sqrt{2}\sinh\sqrt{2}t\}$

19 $y = \frac{1}{5}\{3\sin 2t - 4\cos 2t + \frac{4}{3}\sin 3t + \frac{48}{7}\cos 3t\} - \frac{4}{7}\cos 4t$

20 $x = \cos\left(t\sqrt{\frac{3}{10}}\right) + \frac{3}{4}\cos(t\sqrt{6})$ $y = \frac{5}{4}\cos\left(t\sqrt{\frac{3}{10}}\right) - \frac{1}{4}\cos(t\sqrt{6})$

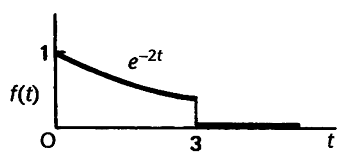
Test exercise 3 (page 109)

1 (a)



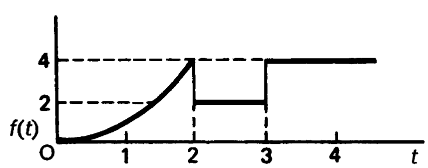
$$F(s) = \frac{3}{s^2}\{1 - e^{-2s}\}$$

(b)



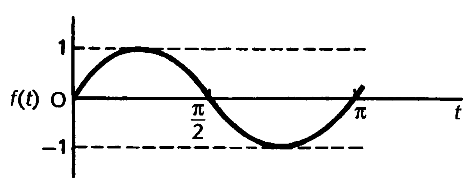
$$F(s) = \frac{1}{s+2}\{1 - e^{-6}e^{-3s}\}$$

(c)



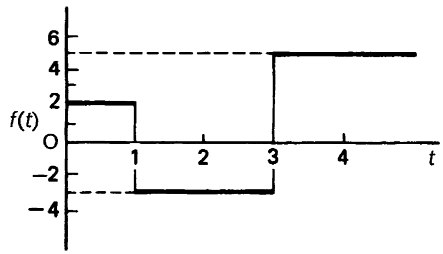
$$F(s) = \frac{2}{s^3} - 2e^{-2s}\left\{\frac{1}{s^3} + \frac{2}{s^2} - \frac{1}{s}\right\} + \frac{2}{s}e^{-3s}$$

(d)

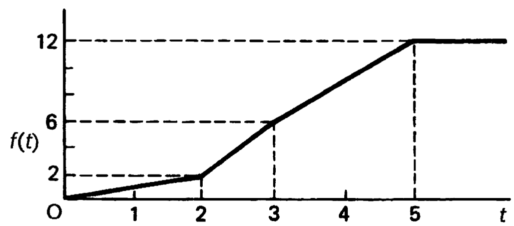


$$F(s) = \frac{2}{s^2+4}\{1 - e^{-\pi s}\}$$

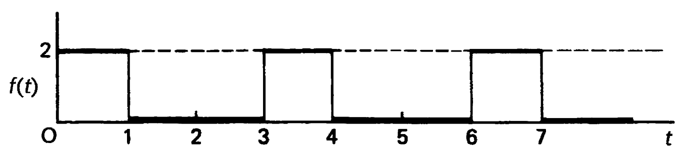
2 $f(t) = 2 \cdot u(t) - 5 \cdot u(t-1) + 8 \cdot u(t-3)$



3 $f(t) = t \cdot u(t) + 3(t-2) \cdot u(t-2) - (t-3) \cdot u(t-3) - 3(t-5) \cdot u(t-5)$



4 $f(t) = 2 \cdot u(t) - 2 \cdot u(t-1) + 2 \cdot u(t-3) - 2 \cdot u(t-4)$
$\quad + 2 \cdot u(t-6) - 2 \cdot u(t-7) + \ldots$

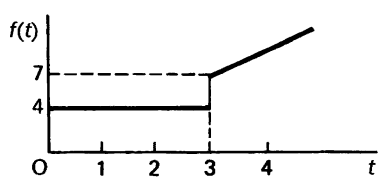


$$\left.\begin{aligned} f(t) &= 2 \quad & 0 < t < 1 \\ &= 0 \quad & 1 < t < 3 \end{aligned}\right\} \qquad f(t) = f(t+3)$$

Further problems 3 (page 110)

1 $f(t) = 3 \cdot u(t) + 2(t-2) \cdot u(t-2) - 2(t-5) \cdot u(t-5)$
2 $f(t) = t \cdot u(t) - (t-1) \cdot u(t-1) + (t-2) \cdot u(t-2) - (t-3) \cdot u(t-3)$

3



$$F(s) = \frac{4}{s} + \frac{3e^{-3s}}{s} + \frac{2e^{-3s}}{s^2}$$

4 (a) $f(t) = t^2 \cdot u(t) - (t^2 - 5t) \cdot u(t-3)$
(b) $f(t) = \cos t \cdot u(t) + (\cos 2t - \cos t) \cdot u(t-\pi) + (\cos 3t - \cos 2t) \cdot u(t - 2\pi)$

5 $F(s) = e^{-2s}\left\{\frac{1}{s^2} + \frac{3}{s}\right\} - e^{-3s}\left\{\frac{1}{s^2} + \frac{4}{s}\right\}$

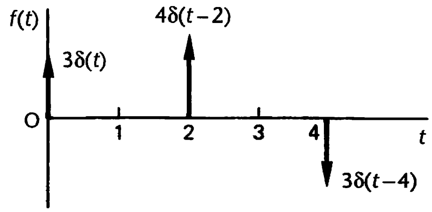
6 (a) $f(t) = t^2 \cdot u(t) - t^2 \cdot u(t-2) + 4 \cdot u(t-2) - 4 \cdot u(t-5)$
(b) $F(s) = \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2} - \frac{4e^{-5s}}{s}$

Test exercise 4 (page 142)

1 $F(s) = \dfrac{2(1 - e^{-2s} - 2se^{-2s})}{s^2(1 - e^{-4s})}$ **2** (a) e^{-6} (b) 0 (c) 11

3 (a) $F(s) = 4e^{-3s}$ (b) $F(s) = e^{-2(3+s)}$

4



$F(s) = 3 + 4e^{-2s} - 3e^{-4s}$

5 $x = e^{-3t}\{4\sin t - \cos t\}$

6 $x = 3e^4 e^{-t} \cdot u(t-4) + e^{-2t}\{2 \cdot u(t) - 3e^8 \cdot u(t-4)\}$

7 (a) $f(t) = \sin t$, frequency 1 radian per unit of time, period 2π units of time

(b) $f(t) = \dfrac{18}{\sqrt{53}} e^{-t/6} \sin\left(\dfrac{\sqrt{53}}{6}\right)t$, frequency $\dfrac{\sqrt{53}}{6}$ radian per unit of time, period $\dfrac{12\pi}{\sqrt{53}}$ units of time

8 Transient solution $\dfrac{e^{-t}}{19}\left(32\sqrt{2}\sin\sqrt{2}t - 40\cos\sqrt{2}t\right)$,

steady-state solution $\dfrac{2}{19}e^{5t}$

Further problems 4 (page 143)

2 $L\{f(t)\} = \dfrac{a(1 + e^{-\pi s})}{(s^2+1)(1 - e^{-\pi s})}$ **3** (a) $F(s) = \dfrac{1}{s^2} - \dfrac{w}{s}\left\{\dfrac{e^{-ws}}{1 - e^{-ws}}\right\}$

(b) $F(s) = \dfrac{1 - e^{2(1-s)\pi}}{(s-1)(1 - e^{-2\pi s})}$ (c) $F(s) = \dfrac{1 - e^{-s}(s+1)}{s^2(1 - e^{-2s})}$

(d) $F(s) = \dfrac{1}{1 - e^{-3s}}\left\{\dfrac{2}{s^3} - \dfrac{2e^{-2s}}{s^3} - \dfrac{4e^{-2s}}{s^2} - \dfrac{4e^{-3s}}{s}\right\}$

4 $x = \dfrac{P}{M\omega}\sin\omega t$ **5** $i = \dfrac{E}{L}\cos\left(\dfrac{t}{\sqrt{LC}}\right)$

6 $x = 2e^{-2t}\{1 + 10e^8 \cdot u(t-4)\} - 2e^{-3t}\{1 + 10e^{12} \cdot u(t-4)\}$

7 (a) $f(t) = 4\sqrt{3}\sin\dfrac{t}{2\sqrt{3}} - \cos\dfrac{t}{2\sqrt{3}}$, frequency $\dfrac{1}{2\sqrt{3}}$ radian per unit of time,

period $4\pi\sqrt{3}$ units of time (b) $f(t) = 2\cos 2\sqrt{3}t - \dfrac{1}{2\sqrt{3}}\sin 2\sqrt{3}t$,

frequency $2\sqrt{3}$ radian per unit of time, period $\pi\sqrt{3}$ units of time

8 (a) $f(t) = -4{\cdot}48\sin 0{\cdot}69t + 1{\cdot}06\cos 0{\cdot}69t$

(b) $f(t) = \dfrac{\pi}{(3/2)^{\frac{1}{4}}}\sin[(1{\cdot}5)^{\frac{1}{4}}t]$

9 Transient solution $e^{-3t/8}\left(\dfrac{421}{9\sqrt{23}}\sin\dfrac{\sqrt{23}}{8}t - \dfrac{1}{9}\cos\dfrac{\sqrt{23}}{8}t\right)$,

steady-state solution $\dfrac{1}{9}e^t$

Test exercise 5 (page 169)

1 $\dfrac{z}{z+1}$ provided $|z|>1$ **2** $-2\dfrac{z^3-4z^2+(2a+1)z}{(z-1)^2(z-a)}$

3 (a) $\dfrac{z(3z-4)}{(z-1)^2}$, $|z|>1$ (b) $\dfrac{25z}{z-5}$, $|z|>5$

4 $\{2k+3-2^{k+1}\}$ **5** $\{3u_k+4k-2^{k+1}\}$ **6** $\dfrac{z\sin T}{z^2-2z\cos T+1}$

Further problems 5 (page 169)

1 $\dfrac{z}{z+a}$ provided $|z|>|a|$ **2** (a) $\left\{\dfrac{1}{12}u_k-\dfrac{3}{4}(-3)^k+\dfrac{2}{3}(-2)^k\right\}$

(b) $\left\{\dfrac{1}{4}u_k-\dfrac{k}{2}+\dfrac{3}{4}(1/3)^k\right\}$ (c) $\left\{\dfrac{2}{3}(3^k)+\dfrac{1}{3}(-3)^k-2k\right\}$

3 $\left\{\dfrac{1}{2}(1+j)(-j)^{k-1}+\dfrac{1}{2}(1-j)(j)^{k-1}\right\}$ **4** (a) $\left\{u_k+\dfrac{3}{2}k(-2)^k\right\}$

(b) $\left\{\dfrac{1}{9}u_k-\dfrac{5}{6}k(-2)^k+\dfrac{8}{9}(-2)^k\right\}$ **5** (a) $\dfrac{z^2}{z^2-1}$ (b) $\dfrac{z}{z^2-1}$

(c) $\dfrac{z^7+z^5+z^4+1}{z^7}$ (d) $\dfrac{z^7+z^6+z^5+z+1}{z^7}$ (e) $\dfrac{z^7+z^6+z^5+z+1}{z^{10}}$

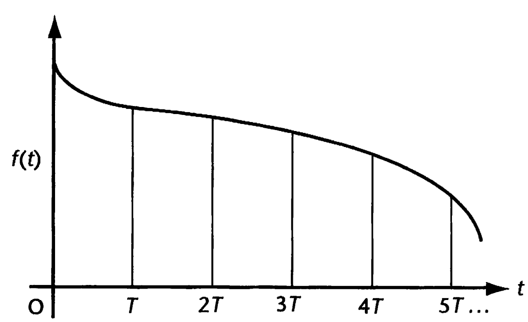
(f) $\dfrac{z^6+z^5+z+1}{z^6}$ **6** (a) $\{x_k\}=\left\{\dfrac{1}{2}((-3)^k-2(-2)^k+(-1)^k)\right\}$ for $k\geq 1$

(b) $\{x_k\}=\left\{\dfrac{1}{2}((-3^{k+1}-(-2)^{k+2}+(-1)^{k+1})\right\}$

(c) $\{x_k\}=\{10(3^k)-7(2^k)\}$ (d) $\{x_k\}=\{6(2^k)-3u_k\}$

9 3 **10** $-\dfrac{2}{7}$ **13** (a) $\dfrac{z\sinh T}{z^2-2z\cosh T+1}$ (b) $\dfrac{z(z-\cosh aT)}{z^2-2z\cosh aT+1}$

(c) $\dfrac{ze^{-aT}(ze^{aT}-\cosh bT)}{z^2-2ze^{-aT}\cosh bT+e^{-2aT}}$



Test exercise 6 (page 227)

1 (a) yes (b) yes (c) no (d) yes (e) no (f) no

2 $f(x)=2\pi-4\{\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ \ldots\}$ **3** (a) odd (b) odd

(c) even (d) neither (e) neither (f) even

4 (a) $f(x)=\dfrac{\pi}{2}+\dfrac{4}{\pi}\left\{\cos x+\dfrac{1}{9}\cos 3x+\dfrac{1}{25}\cos 5x+\ \ldots\right\}$

(b) $f(x) = -2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ \dots\right\}$ **5** (a) cosine terms only
(b) sine terms only; odd harmonics only (c) odd harmonics only
(d) odd harmonics only

6 $f(t) = \frac{1}{2} - \frac{1}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \ \dots\right\}$
$+\frac{1}{\omega}\left\{\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \ \dots\right\}$ where $\omega = \pi/2$

Further problems 6 (page 228)

1 $f(x) = \frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ \dots\right\}$

2 $f(t) = -1 - \frac{16}{\pi}\left\{\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ \dots\right\}$ where $\omega = \frac{\pi}{2}$

3 $f(x) = \frac{4}{\pi}\left\{\frac{1}{2} - \frac{1}{1\times 3}\cos 2x - \frac{1}{3\times 5}\cos 4x - \frac{1}{5\times 7}\cos 6x - \ \dots\right\}$

4 $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ \dots\right\}$

5 $f(x) = \frac{2A}{\pi}\left\{1 - 2\left(\frac{1}{1\times 3}\cos 2x + \frac{1}{3\times 5}\cos 4x + \ \dots\right)\right\}$

6 $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ \dots\right\}$
$+\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ \dots\right\}$

7 $i = f(t) = \frac{A}{\pi}\left\{1 + \frac{\pi}{2}\sin\omega t - 2\left(\frac{1}{1\times 3}\cos 2\omega t + \frac{1}{3\times 5}\cos 4\omega t\right.\right.$
$\left.\left.+\frac{1}{5\times 7}\cos 6\omega t + \ \dots\right)\right\}$ where $\omega = \frac{2\pi}{T}$

8 $f(x) = \frac{3a}{\pi}\left\{\sin 2x + \frac{1}{2}\sin 4x + \frac{1}{4}\sin 8x + \frac{1}{5}\sin 10x + \ \dots\right\}$

9 (a) $f(x) = \frac{\pi^2}{6} - \left(\cos 2x + \frac{1}{4}\cos 4x + \frac{1}{9}\cos 6x + \ \dots\right)$

(b) $f(x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ \dots\right)$

10 $f(x) = \frac{2}{\pi}\left\{\frac{1}{2} + \frac{\pi}{4}\cos x + \frac{1}{1\times 3}\cos 2x - \frac{1}{3\times 5}\cos 4x + \ \dots\right\}$

11 $f(x) = \ -\frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{3\pi}\cos 2x + \frac{2}{15\pi}\cos 4x - \ \dots$

12 $f(x) = \frac{4}{\pi}\left\{\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \ \dots\right\}$

13 $f(t) = -\frac{4}{\pi^2}\left\{\cos\pi t + \frac{1}{9}\cos 3\pi t + \ \dots\right\} + \frac{2}{\pi}\left\{2\sin\pi t - \frac{1}{2}\sin 2\pi t + \ \dots\right\}$

14 $f(x) = \dfrac{\pi^2}{3} - 4\left\{\cos x - \dfrac{1}{4}\cos 2x + \dfrac{1}{9}\cos 3x - \dfrac{1}{16}\cos 4x + \dots\right\}$

15 $f(x) = 7 - \dfrac{6}{\pi}\left\{\sin x - \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x - \dfrac{1}{4}\sin 4x + \dots\right\}$

16 $f(x) = \dfrac{2}{3} + \dfrac{4}{\pi^2}\left\{\cos \pi t - \dfrac{1}{4}\cos 2\pi t + \dfrac{1}{9}\cos 3\pi t - \dots\right\}$

17 $f(x) = -\left\{\sin x + \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x + \dfrac{1}{4}\sin 4x + \dots\right\}$

18 $f(t) = -\dfrac{2}{\pi}\left\{\sin \omega t - \sin 2\omega t + \dfrac{1}{3}\sin 3\omega t + \dfrac{1}{5}\sin 5\omega t + \dots\right\}$ where $\omega = \pi/2$

19 $f(x) = \dfrac{4\pi^2}{3} + 4\left\{\cos x + \dfrac{1}{4}\cos 2x + \dfrac{1}{9}\cos 3x + \dots\right\}$
$$- 4\pi\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right\}$$

20 $f(t) = 1 - 1{\cdot}17\cos \omega t + 0{\cdot}328\cos 2\omega t + 0\cos 3\omega t + \dots$
$+ 0{\cdot}282\sin \omega t + 0{\cdot}288\sin 2\omega t - 0{\cdot}318\sin 3\omega t + \dots$ where $\omega = \pi/3$

Test exercise 7 (page 268)

1 $f(t) = \dfrac{1}{2} + \dfrac{j}{2\pi}\displaystyle\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{e^{j2n\pi t}}{n}$ **2** $F(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{(a - j\omega)\sinh(a + j\omega)}{a^2 + \omega^2}$

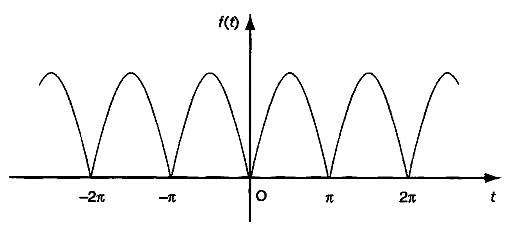
3 $\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sinh a \cos \omega + j \sin \omega \cosh a}{a + j\omega}\right)$ **4** $-\dfrac{j}{2}(F(\omega + \omega_0) - F(\omega - \omega_0))$

5 $2\sqrt{2\pi}(e^t - e^{4t})u(t)$ **6** $F_c(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{k}{k^2\omega^2}$, $F_s(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{\omega}{k^2 + \omega^2}$

Further problems 7 (page 268)

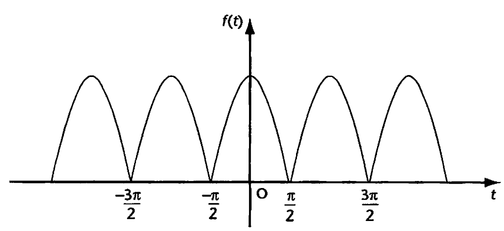
3

$$f(t) = -\frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{4n^2 - 1}e^{2\pi nt}$$



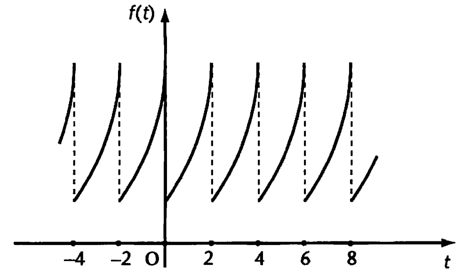
4

$$f(t) = -\frac{4j}{\pi}\sum_{n=-\infty}^{\infty} \frac{n}{4n^2 - 1}e^{j2\pi nt}$$



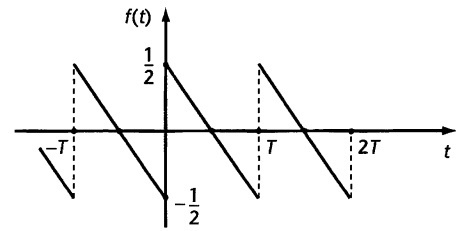
5

$$f(t) = -\frac{e^{2\pi}-1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1+jn}{1+n^2}e^{j\pi nt}$$



6

$$f(t) = \frac{1}{2} + \frac{1}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}e^{j(n\omega_0 t+\pi/2)}$$

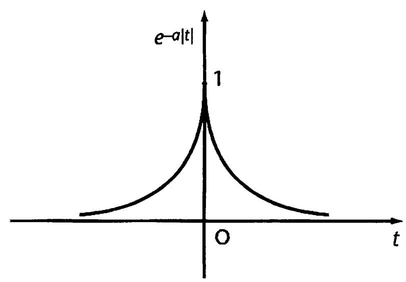


8 $\dfrac{(e^2-1)\cos\omega + \omega(e^2+1)\sin\omega}{\sqrt{2\pi}e(\omega^2+1)}$ **9** $\dfrac{j(\omega(e^2-1)\cos\omega - (e^2+1)\sin\omega)}{\sqrt{2\pi}e(\omega^2+1)}$

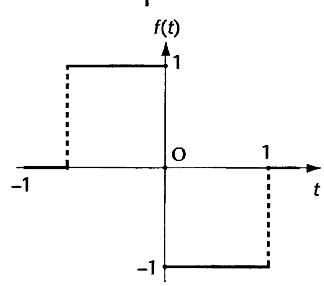
10 $\sqrt{\dfrac{\pi}{2}}\left(\dfrac{1+e^{-j\omega}}{\pi^2-\omega^2}\right)$ **11** $\dfrac{\sqrt{2\pi}\cos(\omega/2)}{\pi^2-\omega^2}$

12

$$F(\omega) = \frac{2a}{a^2+\omega^2}$$



13 (a)



(b) $f(t) = u(t-1) - 2u(t) + u(t+1)$ (c) $F(\omega) = \dfrac{4j}{\omega}\sin^2(\omega/2)$

14 $\dfrac{j}{\sqrt{2\pi}(k^2-\omega^2)}(\omega[1-\cos\pi(k+\omega)] - jk\sin\pi(k-\omega))$

20 $F_s(\omega) = 2\sqrt{\dfrac{2}{\pi}}\dfrac{1}{a^2+\omega^2}\{a\sin\omega\cos a - j\omega\cos\omega\sin a\}$

$F_c(\omega) = 2\sqrt{\dfrac{2}{\pi}}\dfrac{1}{a^2+\omega^2}\{\omega\sin\omega\cos a + ja\cos\omega\sin a\}$

21 $F_s(\omega) = 0 \quad F_c(\omega) = 2\cos\omega\,\mathrm{sinc}\,t$

Test exercise 8 (page 324)

2 $y = a_0\left\{1 + \frac{5x^2}{2} + \frac{15x^4}{8} + \frac{5x^6}{16} + \dots\right\} + a_1\left\{x + \frac{4x^3}{3} + \frac{8x^5}{15} + \dots\right\}$

3 (a) $y = A\left\{1 - \frac{x}{1 \times 2} + \frac{x^2}{(1 \times 2)(2 \times 5)} - \frac{x^3}{(1 \times 2)(2 \times 5)(3 \times 8)} + \dots\right\}$

$+ Bx^{\frac{1}{3}}\left\{1 - \frac{x}{1 \times 4} + \frac{x^2}{(1 \times 4)(2 \times 7)} - \frac{x^3}{(1 \times 4)(2 \times 7)(3 \times 10)} + \dots\right\}$

(b) $y = a_0\left\{1 - \frac{x^4}{3 \times 4} + \frac{x^8}{(3 \times 4)(7 \times 8)} + \dots\right\}$

$+ a_1\left\{x - \frac{x^5}{4 \times 5} + \frac{x^9}{(4 \times 5)(8 \times 9)} + \dots\right\}$

(c) $y_A = A\left\{-\frac{1}{2} - \frac{x}{6} - \dots\right\}$

$y_B = B\left\{\ln x\left(-\frac{1}{2} - \frac{x}{6} - \dots\right) + x^{-2}\left(1 - x + \frac{x^2}{4} + \dots\right)\right\}$ **5** $\frac{1}{3}P_0(x) - \frac{4}{3}P_2(x)$

Further problems 8 (page 324)

1 $y_5 = 64e^{4x}\{16x^3 + 60x^2 + 60x + 15\}$

2 $y_n = (-1)^n e^{-x}\{x^3 - 3nx^2 + n(n-1)3x - n(n-1)(n-2)\}$, $n > 3$

3 $y_4 = 480x + 96$ **4** $y_6 = -\{(x^4 - 180x^2 + 360)\cos x + (24x^3 - 480x)\sin x\}$

5 $y_4 = -4e^{-x}\sin x$ **6** $y_3 = 2x(13 + 12\ln x)$ **8** $y_6 = -1018$

10 (a) $y_{2n} = \{x^2 + 2n(2n-1)\}\sinh x + 4nx\cosh x$

(b) $y_{2n} = \{x^3 + 6n(2n-1)x\}\cosh x + \{6nx^2 + 2n(2n-1)(2n-2)\}\sinh x$

11 $y_6 = 2^5 e^{2x}\{2x^3 + 24x^2 + 81x + 75\}$ **12** $y_3 = 2\sqrt{2}a^3 e^{-ax}\{\cos(ax + \pi/4)\}$

14 $y = y_0\left\{1 + \frac{9x^2}{2} + \frac{15x^4}{8} - \frac{7x^6}{16} + \frac{27x^8}{128} + \dots\right\} + y_1\left\{x + \frac{4x^3}{3}\right\}$

15 $y = A(1 + x^2) + Be^{-x}$

16 $y = y_0\left\{1 + \frac{3^2 \times x^2}{2!} + \frac{3^2 \times 5^2 \times x^4}{4!} + \frac{3^2 \times 5^2 \times 7^2 \times x^6}{6!} + \dots\right\}$

$+ y_1\left\{x + \frac{4^2 \times x^3}{3!} + \frac{4^2 \times 6^2 \times x^5}{5!} + \dots\right\}$

17 $y = y_1 x + y_0\left\{1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \dots\right\}$

18 $y = y_0\left\{1 - \frac{2x}{2^2} + \frac{2^2 \times x^4}{2^2 \times 4^2} - \frac{2^3 \times x^6}{2^2 \times 4^2 \times 6^2} + \dots\right\}$

$+ y_1\left\{x - \frac{2x^3}{3^2} + \frac{2^2 \times x^5}{3^2 \times 5^2} - \frac{2^3 \times x^7}{3^2 \times 5^2 \times 7^2} + \dots\right\}$

19 $y = A\left\{1 + x + \frac{x^2}{2 \times 4} + \frac{x^3}{(2 \times 3)(4 \times 7)} + \frac{x^4}{(2 \times 3 \times 4)(4 \times 7 \times 10)} + \dots\right\}$

$+ Bx^{\frac{2}{3}}\left\{1 + \frac{x}{1 \times 5} + \frac{x^2}{(1 \times 2)(5 \times 8)} + \frac{x^3}{(1 \times 2 \times 3)(5 \times 8 \times 11)} + \dots\right\}$

20 $y = a_0\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right\} + a_1\left\{x - \frac{x^3}{3!} + \dots\right\}$

21 $y = a_0\left\{1 + \frac{x^3}{2 \times 3} + \frac{x^6}{(2 \times 3)(5 \times 6)} + \dots\right\}$

$$+ a_1\left\{x + \frac{x^4}{3 \times 4} + \frac{x^7}{(3 \times 4)(6 \times 7)} + \dots\right\}$$

22 $y = A\left\{1 - \frac{x}{1 \times 4} + \frac{x^2}{(1 \times 2)(4 \times 7)} - \frac{x^3}{(1 \times 2 \times 3)(4 \times 7 \times 10)} + \dots\right\}$

$$+ Bx^{-\frac{1}{3}}\left\{1 - \frac{x}{1 \times 2} + \frac{x^2}{(1 \times 2)(2 \times 5)} - \frac{x^3}{(1 \times 2 \times 3)(2 \times 5 \times 8)} + \dots\right\}$$

23 $y = a_1 x + a_0\left\{1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{3x^6}{6!} - \frac{(3)(5)x^8}{8!} + \dots\right\}$

24 $y = u + v$ where $u = A\left\{\frac{-x^4}{4!\,3!} + \frac{x^5}{5!\,3!} - \dots\right\}$

$$v = B\left\{\ln x\left(\frac{-x^4}{4!\,3!} + \frac{x^5}{5!\,3!} - \dots\right) + \left(1 + \frac{x}{1 \times 3} + \frac{x^2}{(1 \times 2)(2 \times 3)} + \dots\right)\right\}$$

25 $y = u + v$ where $u = A\left\{1 + \frac{3x}{1^2} + \frac{3^2 \times x^2}{1^2 \times 2^2} + \frac{3^3 \times x^3}{1^2 \times 2^2 \times 3^2} + \dots\right\}$

$$v = B\left\{\ln x\left(1 + \frac{3x}{1^2} + \frac{3^2 \times x^2}{1^2 \times 2^2} + \frac{3^3 \times x^3}{1^2 \times 2^2 \times 3^2} + \dots\right)\right.$$
$$\left. - \left(\frac{2 \times 3x}{1^2} + \frac{3 \times 3^2 \times x^2}{1^2 \times 2^2} + \frac{11 \times 3^3 \times x^3}{1^2 \times 2^2 \times 3^3} + \dots\right)\right\}$$

26 eigenfunctions: $y_n(x) = A_n \cos\sqrt{\lambda_n}x$; eigenvalues: $\lambda_n = \frac{(2n+1)^2\pi^2}{4}$

27 $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$, $H_3 = 8x^3 - 12x$

28 $L_0 = 1$, $L_1 = 1 - x$, $L_2 = 2 - 4x + x^2$, $L_3 = 6 - 18x + 9x^2 - x^3$

Text exercise 9 (page 367)

1

x	y
0	1·0
0·1	1·1
0·2	1·211
0·3	1·3352
0·4	1·4753
0·5	1·6343

2

x	y
1	0
1·2	0·204
1·4	0·4211
1·6	0·6600
1·8	0·9264
2·0	1·2243

3

x	y
0	1·0
0·1	1·2052
0·2	1·4214
0·3	1·6499
0·4	1·8918
0·5	2·1487

4

x	y
2·0	3·0
2·1	3·005
2·2	3·0195
2·3	3·0427
2·4	3·0736
2·5	3·1117

5

x	y
1·0	0
1·1	0·1052
1·2	0·2215
1·3	0·3401
1·4	0·4717
1·5	0·6180

6

x	y
0·0	1·0000
0·1	1·0101
0·2	1·0202
0·3	1·0305
0·4	1·0408
0·5	1·0513
0·6	1·0619
0·7	1·0726
0·8	1·0834
0·9	1·0943
1·0	1·1053

Further problems 9 (page 368)

1

x	y
0	1·0
0·2	0·8
0·4	0·72
0·6	0·736
0·8	0·8288
1·0	0·9830

2

x	y
0	1·4
0·1	1·596
0·2	0·8707
0·3	2·2607
0·4	2·8318
0·5	3·7136

3

x	y
1·0	2·0
1·2	2·0333
1·4	2·1143
1·6	2·2250
1·8	2·3556
2·0	2·5000

4

x	y
0	0·5
0·1	0·543
0·2	0·5716
0·3	0·5863
0·4	0·5878
0·5	0·5768

5

x	y
0	1·0
0·1	1·1022
0·2	1·2085
0·3	1·3179
0·4	1·4296
0·5	1·5428

6

x	y
1·0	1·0
1·1	1·1871
1·2	1·3531
1·3	1·5033
1·4	1·6411
1·5	1·7688

7

x	y
0	0
0·1	0·1002
0·2	0·2015
0·3	0·3048
0·4	0·4110
0·5	0·5214

8

x	y
0	1·0
0·2	0·8562
0·4	0·8110
0·6	0·8465
0·8	0·9480
1·0	1·1037

9

x	y
0	1·0
0·1	0·9138
0·2	0·8512
0·3	0·8076
0·4	0·7798
0·5	0·7653

10

x	y
0	0·4
0·2	0·4259
0·4	0·4374
0·6	0·4319
0·8	0·4085
1·0	0·3689

11

x	y
1·0	2·0
1·2	2·4197
1·4	2·8776
1·6	3·3724
1·8	3·9027
2·0	4·4677

12

x	y
0	1·0
0·2	1·1997
0·4	1·3951
0·6	1·5778
0·8	1·7358
1·0	1·8540

13

x	y
0	1·0
0·2	1·1679
0·4	1·2902
0·6	1·3817
0·8	1·4497
1·0	1·4983

14

x	y
0	1·0
0·1	1·11
0·2	1·2422
0·3	1·4013
0·4	1·5937
0·5	1·8271

15

x	y
0	3·0
0·1	2·88
0·2	2·5224
0·3	1·9368
0·4	1·1424
0·5	0·1683

16

x	y
0	0
0·2	0·1987
0·4	0·3897
0·6	0·5665
0·8	0·7246
1·0	0·8624

17

x	y
0	1·0
0·2	1·1972
0·4	1·3771
0·6	1·5220
0·8	1·6161
1·0	1·6487

18

x	y
0	2·0
0·1	2·0845
0·2	2·1367
0·3	2·1554
0·4	2·1407
0·5	2·0943

19

x	y
0	1·0
0·2	1·0367
0·4	1·1373
0·6	1·2958
0·8	1·5145
1·0	1·8029

20

x	y
1·0	0
1·2	0·1833
1·4	0·3428
1·6	0·4875
1·8	0·6222
2·0	0·7500

21

x	y
1·0	2·0000
1·2	2·0333
1·4	2·1121
1·6	2·2219
1·8	2·3522
2·0	2·4965

22

x	y
0·0	1·0000
0·2	0·8600
0·4	0·8118
0·6	0·8452
0·8	0·9454
1·0	1·1002

23

x	y
1·0	2·0000
1·2	2·4191
1·4	2·8769
1·6	3·3715
1·8	3·9018
2·0	4·4666

Test exercise 10 (page 411)

2 $145{\cdot}7 \pm 2{\cdot}6$ mm **3** $5{\cdot}8$ m/s **4** $\dfrac{-2(x+y)}{2x+3y}$; $\dfrac{-2}{(2x+3y)^3}$

5 $\dfrac{x}{2(x^2-y^2)}$; $\dfrac{-y}{4(x^2-y^2)}$; $\dfrac{-y}{2(x^2-y^2)}$; $\dfrac{x}{4(x^2-y^2)}$

6 (a) $(-1, 1)$, saddle; $(-1, -\frac{4}{3})$, min (b) an infinity of maxima along the line $y = 5x/2$ when $z = 4$ **7** $1{\cdot}10$ m $\times$ $1{\cdot}10$ m $\times$ $0{\cdot}825$ m high

8 $u = \dfrac{8}{7}$, $x = \dfrac{6}{7}$, $y = -\dfrac{4}{7}$, $z = \dfrac{2}{7}$

Further problems 10 (page 412)

1 $(8x\cos x - 6y\sin x)/J$; $-(4x^3\cos y + 6x\sin y)/J$;
$J = 4x\cos x\sin y + 2x^2 y\sin x\cos y$ **2** $e^{3y}/2(xe^{3y}+e^{-3y})$; $e^{-3y}/2(xe^{3y}+e^{-3y})$;
$-1/3(xe^{3y}+e^{-3y})$; $x/3(xe^{3y}+e^{-3y})$

5 $(2e^{-x}\sinh 2x\sin 3y + 3ye^{-x}\cosh 2x\cos 3y)/(1+3y^2)$;
$\{-4ye^{x}\sinh 2x\sin 3y + 3e^{x}(1+y^2)\cosh 2x\cos 3y\}/2(1+3y^2)$

7 (a) $(4, -4, -11)$, min (b) $(1, -2, 4)$, saddle (c) $(\frac{10}{7}, \frac{6}{7}, \frac{97}{7})$, max

8 (0, 0), saddle; (2, 0), min; (−2, 0), min **9** (2, 1), max; $(-\frac{2}{3}, -\frac{1}{3})$, min

10 (0, 0); (3, 3); (−3, −3), all saddle points

11 (a) (1, 0), saddle; (1, 1), min; $(-2, \frac{1}{2})$, saddle; $(-\frac{7}{5}, \frac{1}{5})$, max
(b) (0, 0), max; (1, 1); (1, −1); (−1, 1); (−1, −1), all four saddle points

12 (a) A point of inflexion at the origin (b) An infinity of maxima along the line $y = x/4$ when $z = 6$ (c) The value of z ranges from −1 to 1 and has an infinity of stationary points lying on the circles $x^2 + y^2 = n\pi$. When n is even the stationary points are maxima and when n is odd the stationary points are minima. There is also a single maximum at (0, 0, 1)

13 $x = 66{\cdot}7$ mm; $\theta = \dfrac{\pi}{3}$ **14** $l = h = \dfrac{1}{5\pi}\sqrt[3]{60\pi^2 V}$; $d = l\sqrt{5}$ **15** $l = 1{\cdot}00$ cm;
$d = 4.48$ cm; $\theta = 48°\ 11'$ **16** cube of side $\dfrac{2r}{\sqrt{3}}$; $V_{max} = \dfrac{8r^3}{3\sqrt{3}}$

17 (a) $u = \dfrac{64}{27}$; $x = y = z = \pm\dfrac{2}{\sqrt{3}}$; $u = \dfrac{9}{7}$, $x = y = \pm\dfrac{3}{\sqrt{14}}$ (b) $u = 9$;
$x = \pm\dfrac{3}{\sqrt{2}}$, $y = \mp\dfrac{3}{\sqrt{2}}$

Test exercise 11 (page 448)

1 (a) $u = 2x^4(t-2) + 4xt + e^{2t}$ (b) $u = 2\sin 2x\cdot(e^y - 1) + \sin x + y^2$

2 $u(x,t) = \dfrac{16}{\pi^2}\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^2}\cdot\sin\frac{r\pi}{2}\cdot\sin\frac{r\pi x}{10}\cdot\cos\frac{r\pi t}{10}$

3 $u(x,t) = \dfrac{100}{\pi}\displaystyle\sum_{r=1}^{\infty}(-1)^{r+1}\cdot\frac{1}{r}\sin\frac{\lambda x}{c}\cdot e^{-\lambda^2 t}$ where $\lambda = \dfrac{r\pi c}{2}$

4 $u(x,y) = \displaystyle\sum_{r=1}^{\infty}\frac{20}{r\pi}\cdot\operatorname{cosech} r\pi\cdot\sin\frac{r\pi x}{2}\cdot\sinh\frac{r\pi y}{2}$ with $r = 1, 3, 5, \ldots$

5 $v(r, \theta) = 5r^3\cos 3\theta$

Further problems 11 (page 449)

2 $u(x,t) = \dfrac{32}{\pi^3}\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^3}\cdot\sin\frac{r\pi x}{2}\cdot\cos\frac{3r\pi t}{2}$ (r odd)

3 $u(x,t) = \dfrac{2}{25\pi^2}\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^2}\cdot\sin\frac{r\pi}{2}\cdot\sin\frac{r\pi x}{4}\cdot\cos\frac{5r\pi t}{2}$

4 $u(x,t) = \dfrac{25}{2\pi^2}\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^2}\cdot\sin\frac{r\pi}{5}\cdot\sin\frac{r\pi x}{10}\cdot\cos\frac{cr\pi t}{10}$

5 $u(x,t) = \dfrac{800}{\pi^3}\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^3}\cdot\sin\frac{r\pi x}{10}\cdot e^{-4\lambda^2 t}$ with $r = 1, 3, 5, \ldots$ where $\lambda = \dfrac{r\pi}{10}$

6 $u(x,t) = \dfrac{16}{\pi^2}\sum_{r=1}^{\infty}\dfrac{1}{r^2}\cdot\sin\dfrac{r\pi}{2}\cdot\sin\dfrac{r\pi x}{10}\cdot e^{-r^2c^2\pi^2t/100}$ with $r = 1, 3, 5, \ldots$

7 $u(x,y) = \dfrac{128}{\pi^3}\sum_{r=1}^{\infty}\dfrac{1}{r^3}\cdot\operatorname{cosech}\dfrac{r\pi}{2}\cdot\sinh\dfrac{r\pi}{4}(2-y)\cdot\sin\dfrac{r\pi x}{4}$ with $r = 1, 3, 5, \ldots$

8 $u(x,y) = \dfrac{200}{\pi^3}\sum_{r=1}^{\infty}\dfrac{1}{r^3}\cdot\operatorname{cosech}\dfrac{2r\pi}{5}\cdot\sin\dfrac{r\pi x}{5}\cdot\sinh\dfrac{r\pi}{5}(y-2)$ with $r = 1, 3, 5, \ldots$

9 $v(r,\theta) = -4r\cos\theta + r^2\sin 2\theta$ **10** $v(r,\theta) = 3(1 - r^2\cos 2\theta)$

Test exercise 12 (page 513)

1 (a) solutions unique (b) infinite number of solutions **2** $x_1 = -4$, $x_2 = 2$, $x_3 = -3$ **3** $x_1 = -2$, $x_2 = 2$, $x_3 = 3$ **4** $x_1 = -3$, $x_2 = 4$, $x_3 = -2$

5 $x_1 = 1$, $x_2 = -2$, $x_3 = 2$ **6** $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$. $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

$x_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$; $x_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ **7** $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$; $\mathbf{M}^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$

$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$ **8** $f_1(x) = -\dfrac{10}{3}e^{6x} + \dfrac{1}{3}e^{3x}$; $f_2(x) = \dfrac{5}{3}e^{6x} + \dfrac{1}{3}e^{3x}$

9 $f_1(x) = \dfrac{1}{3}\cos\sqrt{5}x + \dfrac{4}{3\sqrt{5}}\sin\sqrt{5}x + \dfrac{2}{3}\cosh 2x + \dfrac{1}{3}\sinh 2x$

$f_2(x) = -\dfrac{1}{3}\cos\sqrt{5}x - \dfrac{4}{3\sqrt{5}}\sin\sqrt{5}x + \dfrac{1}{3}\cosh 2x + \dfrac{1}{6}\sinh 2x$

10 (a) $\begin{bmatrix} -8 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7{\cdot}196 \\ -0{\cdot}464 \end{bmatrix}$

Further problems 12 (page 514)

1 $x_1 = 1$, $x_2 = -4$, $x_3 = 3$ **2** (a) $x_1 = 3$, $x_2 = 1$, $x_3 = -4$ (b) $x_1 = 4$, $x_2 = -2$, $x_3 = -1$ **3** (a) $x_1 = 4$, $x_2 = 2$, $x_3 = 5$, $x_4 = 3$ (b) $x_1 = 5$, $x_2 = -4$, $x_3 = 1$, $x_4 = 3$ (c) $x_1 = 3$, $x_2 = -2$, $x_3 = 0$, $x_4 = 5$

4 (a) $x_1 = -3$, $x_2 = 1$, $x_3 = 3$ (b) $x_1 = 5$, $x_2 = 2$, $x_3 = -1$ (c) $x_1 = 4$, $x_2 = 3$, $x_3 = -1$, $x_4 = -2$ **5** (a) $\lambda_1 = 2$, $\lambda_2 = 7$; $x_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$; $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $\lambda_1 = 1$, $\lambda_2 = -3$; $x_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$; $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\lambda_1 = -8$, $\lambda_2 = 4$; $x_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$; $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) $\lambda_1 = 4$, $\lambda_2 = -6$; $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $x_2 = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$ (e) $\lambda_1 = 1$, $\lambda_2 = 3$; $\lambda_3 = 9$;

$x_1 = \begin{bmatrix} 7 \\ -1 \\ -5 \end{bmatrix}$; $x_2 = \begin{bmatrix} 7 \\ 1 \\ -7 \end{bmatrix}$; $x_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ (f) $\lambda = 1, 2, 4$; $x = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

(g) $\lambda = -1, -3, 7$; $x = \begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 27 \\ 10 \end{bmatrix}$ (h) $\lambda = -2, 4, 7$;

$x = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$

6 (a) $f_1(x) = \frac{1}{4}(5e^x - e^{-3x})$; $f_2(x) = \frac{1}{4}(e^x - e^{-3x})$

(b) $f_1(x) = \frac{9}{5}(e^{-6x} - e^{4x})$; $f_2(x) = -\frac{1}{5}(e^{-6x} + 9e^{4x})$

(c) $f_1(x) = \frac{1}{2}(5e^{4x} - 3e^{2x})$; $f_2(x) = \frac{2}{3}e^x - \frac{3}{2}e^{2x} + \frac{5}{6}e^{4x}$;

$f_3(x) = 4e^x - \frac{9}{2}e^{2x} + \frac{5}{2}e^{4x}$ (d) $f_1(x) = 3e^{-2x} - e^{4x} + 2e^{7x}$;

$f_2(x) = -e^{-2x} - \frac{4}{3}e^{4x} + \frac{1}{3}e^{7x}$; $f_3(x) = e^{-2x} - \frac{5}{3}e^{4x} - \frac{1}{3}e^{7x}$ **7** $\lambda = 0, 7, 13$

8 $I_1 = 2, I_2 = -3, I_3 = 2$ **9** $k = 2$; $x_1 = -2$, $x_2 = \frac{1}{2}$, $x_3 = 1$

10 (a) $f_1(x) = \frac{3}{5}\cosh\sqrt{2}x + \frac{9}{5\sqrt{2}}\sinh\sqrt{2}x + \frac{2}{5}\cosh\sqrt{7}x + \frac{11}{5\sqrt{7}}\sinh\sqrt{7}x$;

$f_2(x) = -\frac{2}{5}\cosh\sqrt{2}x - \frac{6}{5\sqrt{2}}\sinh\sqrt{2}x + \frac{2}{5}\cosh\sqrt{7}x + \frac{11}{5\sqrt{7}}\sinh\sqrt{7}x$

(b) $f_1(x) = -\frac{5}{12}\cos 2\sqrt{2}x + \frac{5}{12\sqrt{2}}\sin 2\sqrt{2}x + \frac{5}{12}\cosh 2x + \frac{1}{12}\sinh 2x$;

$f_2(x) = \frac{1}{6}\cos 2\sqrt{2}x - \frac{1}{6\sqrt{2}}\sin 2\sqrt{2}x + \frac{5}{6}\cosh 2x + \frac{1}{6}\sinh 2x$

(c) $f_1(x) = -\frac{35}{16}\cosh x + \frac{7}{16}\sinh x + \frac{35}{12}\cosh\sqrt{3}x - \frac{7}{12\sqrt{3}}\sinh\sqrt{3}x$

$+\frac{13}{48}\cosh 3x + \frac{7}{144}\sinh 3x$; $f_2(x) = \frac{5}{16}\cosh x - \frac{1}{16}\sinh x + \frac{5}{12}\cosh\sqrt{3}x$

$-\frac{1}{12\sqrt{3}}\sinh\sqrt{3}x + \frac{13}{48}\cosh 3x + \frac{7}{144}\sinh 3x$; $f_3(x) = \frac{25}{16}\cosh x - \frac{5}{16}\sinh x$

$-\frac{35}{12}\cosh\sqrt{3}x + \frac{7}{12\sqrt{3}}\sinh\sqrt{3}x + \frac{65}{48}\cosh 3x + \frac{35}{144}\sinh 3x$

(d) $f_1(x) = -\frac{9}{8}\cos x + \frac{9}{4}\sin x + \frac{19}{10}\cos\sqrt{3}x - \frac{12}{5\sqrt{3}}\sin\sqrt{3}x + \frac{9}{40}\cosh\sqrt{7}x$

$+\frac{3}{20\sqrt{7}}\sinh\sqrt{7}x$; $f_2(x) = \frac{15}{16}\cos x - \frac{15}{8}\sin x - \frac{19}{20}\cos\sqrt{3}x + \frac{6}{5\sqrt{3}}\sin\sqrt{3}x$

$+\frac{81}{80}\cosh\sqrt{7}x + \frac{27}{40\sqrt{7}}\sinh\sqrt{7}x$; $f_3(x) = -\frac{3}{8}\cos x + \frac{3}{4}\sin x + \frac{3}{8}\cosh\sqrt{7}x$

$+\frac{1}{4\sqrt{7}}\sinh\sqrt{7}x$

Test exercise 13 (page 560)

1 $f(1/4, 1/3) = -19/12$, $f(1/2, 1/3) = -5/6$, $f(3/4, 1/3) = -1/12$,
$f(1/4, 2/3) = 1/12$, $f(1/2, 2/3) = 5/6$, $f(3/4, 2/3) = 19/12$

2 $f(1/3, 1/3) = 4$, $f(2/3, 1/3) = 17/3$, $f(1, 1/3) = 26/3$, $f(1/3, 2/3) = 2/3$,
$f(2/3, 2/3) = 3$, $f(1, 2/3) = 16/3$ **3** (a) parabolic (b) hyperbolic
(c) parabolic (d) hyperbolic (e) elliptic **4** $f(1/3, 1/3) = -1{\cdot}61728$,
$f(2/3, 1/3) = -1{\cdot}18519$, $f(1, 1/3) = -0{\cdot}82716$, $f(1/3, 2/3) = -1{\cdot}61728$,
$f(2/3, 2/3) = -1{\cdot}18519$, $f(1, 2/3) = -0{\cdot}82716$

5

t\x	0·0	0·2	0·4	0·6	0·8	1·0	1·2
0·00	0·00000	0·04000	0·16000	0·36000	0·64000	1·00000	0·89000
0·02	0·00000	0·08000	0·20000	0·40000	0·68000	0·76500	0·93000
0·04	0·00000	0·10000	0·24000	0·44000	0·58250	0·80500	0·83250
0·06	0·00000	0·12000	0·27000	0·41125	0·62250	0·70750	0·87250
0·08	0·00000	0·13500	0·26563	0·44625	0·55938	0·74750	0·80938
0·10	0·00000	0·13281	0·29063	0·41250	0·59688	0·68438	0·84688
0·12	0·00000	0·14531	0·27266	0·44375	0·54844	0·72188	0·79844
0·14	0·00000	0·13633	0·29453	0·41055	0·58281	0·67344	0·83281
0·16	0·00000	0·14727	0·27344	0·43867	0·54199	0·70781	0·79199

6

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·000	1·000000	0·840000	0·760000	0·760000	0·840000	1·000000
0·040	1·000000	0·898182	0·832727	0·832727	0·898182	1·000000
0·080	1·000000	0·929917	0·886942	0·886942	0·929917	1·000000
0·120	1·000000	0·952517	0·923125	0·923125	0·952517	1·000000
0·160	1·000000	0·967729	0·94779	0·94779	0·967729	1·000000
0·200	1·000000	0·978081	0·964533	0·964533	0·978081	1·000000

Further problems 13 (page 561)

1

x\y	0·00	0·33	0·67	1·00
0·00	−3·0000	−2·3333	−1·6667	−1·0000
0·25	−2·7500	−2·0833	−1·4167	−0·7500
0·50	−2·5000	−1·8333	−1·1667	−0·5000
0·75	−2·2500	−1·5833	−0·9167	−0·2500
1·00	−2·0000	−1·3333	−0·6667	0·0000

2

x\y	0·00	0·33	0·67	1·00
0·00	4·0000	7·3333	10·6667	14·0000
0·33	6·3333	9·6667	13·0000	16·3333
0·67	8·6667	12·0000	15·3333	18·6667
1·00	11·0000	14·3333	17·6667	21·0000

3

x\y	0·00	0·33	0·67	1·00
0·00	−1·0000	−1·0000	−1·0000	−1·0000
0·33	−0·6667	−0·7500	−0·8000	−0·8333
0·67	−0·3333	−0·5000	−0·6000	−0·6667
1·00	0·0000	−0·2500	−0·4000	−0·5000

4

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·25	0·0000	−0·0069	−0·0694	−0·1875
0·50	0·0000	0·0278	−0·0556	−0·2500
0·75	0·0000	0·1042	0·0417	−0·1875
1·00	0·0000	0·2222	0·2222	0·0000

5

x\y	0·00	0·33	0·67	1·00
0·00	15·0000	16·6667	18·3333	20·0000
0·33	17·3333	19·0000	20·6667	22·3333
0·67	19·6667	21·3333	23·0000	24·6667
1·00	22·0000	23·6667	25·3333	27·0000

6

x\y	0·00	0·33	0·67	1·00
0·00	21·0000	20·0000	19·0000	18·0000
0·33	22·6667	21·6667	20·6667	19·6667
0·67	24·3333	23·3333	22·3333	21·3333
1·00	26·0000	25·0000	24·0000	23·0000

7

x\y	0·00	0·33	0·67	1·00
0·00	4·0000	4·0000	4·0000	4·0000
0·33	4·2222	4·1111	3·7778	3·2222
0·67	4·8889	4·6667	4·0000	2·8889
1·00	6·0000	5·6667	4·6667	3·0000

8

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·0000	0·0000	−0·0741	−0·2963
0·67	0·0000	0·0741	0·0000	−0·3704
1·00	0·0000	0·2963	0·3704	0·0000

9

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	−0·5556	−2·2222	−5·0000
0·33	0·3333	−0·2222	−1·8889	−4·6667
0·67	1·3333	0·7778	−0·8889	−3·6667
1·00	3·0000	2·4444	0·7778	−2·0000

10

x\y	0·00	0·33	0·67	1·00
0·00	−1·0000	−1·0000	−1·0000	−1·0000
0·33	−1·0000	−0·7037	−0·3333	0·1111
0·67	−1·0000	−0·3333	0·4815	1·4444
1·00	−1·0000	0·1111	1·4444	3·0000

11

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·1111	0·1050	0·0873	0·0600
0·67	0·4444	0·4200	0·3493	0·2401
1·00	1·0000	0·9450	0·7859	0·5403

12

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0370	0·2963	1·0000
0·33	0·0370	0·1481	0·5556	1·4815
0·67	0·2963	0·5556	1·1852	2·4074
1·00	1·0000	1·4815	2·4074	4·0000

13

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·0000	0·1111	0·2222	0·3333
0·67	0·0000	0·2222	0·4444	0·6667
1·00	0·0000	0·3333	0·6667	1·0000

14

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·0000	0·0000	−0·0741	−0·2222
0·67	0·0000	0·0741	0·0000	−0·2222
1·00	0·0000	0·2222	0·2222	0·0000

15

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	0·0000	−0·1600	−0·2400	−0·2400	−0·1600	0·0000
0·02	0·0400	−0·1200	−0·2000	−0·2000	−0·1200	0·0400
0·04	0·0800	−0·0800	−0·1600	−0·1600	−0·0800	0·0800
0·06	0·1200	−0·0400	−0·1200	−0·1200	−0·0400	0·1200
0·08	0·1600	0·0000	−0·0800	−0·0800	0·0000	0·1600
0·10	0·2000	0·0400	−0·0400	−0·0400	0·0400	0·2000
0·12	0·2400	0·0800	0·0000	0·0000	0·0800	0·2400
0·14	0·2800	0·1200	0·0400	0·0400	0·1200	0·2800
0·16	0·3200	0·1600	0·0800	0·0800	0·1600	0·3200
0·18	0·3600	0·2000	0·1200	0·1200	0·2000	0·3600
0·20	0·4000	0·2400	0·1600	0·1600	0·2400	0·4000

16

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	0·0000	0·1987	0·3894	0·5646	0·7174	0·8415
0·02	0·0000	0·1983	0·3886	0·5635	0·7159	0·8398
0·04	0·0000	0·1979	0·3879	0·5624	0·7145	0·8381
0·06	0·0000	0·1975	0·3871	0·5613	0·7131	0·8364
0·08	0·0000	0·1971	0·3863	0·5601	0·7116	0·8348
0·10	0·0000	0·1967	0·3855	0·5590	0·7102	0·8331
0·12	0·0000	0·1963	0·3848	0·5579	0·7088	0·8314
0·14	0·0000	0·1959	0·3840	0·5568	0·7074	0·8298
0·16	0·0000	0·1955	0·3832	0·5557	0·7060	0·8281
0·18	0·0000	0·1951	0·3825	0·5546	0·7046	0·8265
0·20	0·0000	0·1947	0·3817	0·5535	0·7032	0·8248

17

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	0·0000	0·3830	0·7596	1·1239	1·4698	1·7916
0·02	0·0000	0·3798	0·7534	1·1147	1·4578	1·7770
0·04	0·0000	0·3767	0·7473	1·1056	1·4459	1·7624
0·06	0·0000	0·3736	0·7412	1·0966	1·4341	1·7481
0·08	0·0000	0·3706	0·7351	1·0876	1·4223	1·7338
0·10	0·0000	0·3676	0·7291	1·0787	1·4107	1·7196
0·12	0·0000	0·3646	0·7232	1·0699	1·3992	1·7056
0·14	0·0000	0·3616	0·7173	1·0612	1·3878	1·6916
0·16	0·0000	0·3586	0·7114	1·0525	1·3764	1·6778
0·18	0·0000	0·3557	0·7056	1·0439	1·3652	1·6641
0·20	0·0000	0·3528	0·6998	1·0354	1·3541	1·6505

18

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	−1·0000	−0·7600	−0·4400	−0·0400	0·4400	1·0000
0·04	−0·9200	−0·6800	−0·3600	0·0400	0·5200	1·0800
0·08	−0·8400	−0·6000	−0·2800	0·1200	0·6000	1·1600
0·12	−0·7600	−0·5200	−0·2000	0·2000	0·6800	1·2400
0·16	−0·6800	−0·4400	−0·1200	0·2800	0·7600	1·3200
0·20	−0·6000	−0·3600	−0·0400	0·3600	0·8400	1·4000
0·24	−0·5200	−0·2800	0·0400	0·4400	0·9200	1·4800
0·28	−0·4400	−0·2000	0·1200	0·5200	1·0000	1·5600
0·32	−0·3600	−0·1200	0·2000	0·6000	1·0800	1·6400
0·36	−0·2800	−0·0400	0·2800	0·6800	1·1600	1·7200
0·40	−0·2000	0·0400	0·3600	0·7600	1·2400	1·8000
0·44	−0·1200	0·1200	0·4400	0·8400	1·3200	1·8800
0·48	−0·0400	0·2000	0·5200	0·9200	1·4000	1·9600
0·52	0·0400	0·2800	0·6000	1·0000	1·4800	2·0400
0·56	0·1200	0·3600	0·6800	1·0800	1·5600	2·1200
0·60	0·2000	0·4400	0·7600	1·1600	1·6400	2·2000

19

t\x	0·00	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00
0·00	0·0000	−0·9000	−1·6000	−2·1000	−2·4000	−2·5000	−2·4000	−2·1000	−1·6000	−0·9000	0·0000
0·02	0·4000	−0·5000	−1·2000	−1·7000	−2·0000	−2·1000	−2·0000	−1·7000	−1·2000	−0·5000	0·4000
0·04	0·8000	−0·1000	−0·8000	−1·3000	−1·6000	−1·7000	−1·6000	−1·3000	−0·8000	−0·1000	0·8000
0·06	1·2000	0·3000	−0·4000	−0·9000	−1·2000	−1·3000	−1·2000	−0·9000	−0·4000	0·3000	1·2000
0·08	1·6000	0·7000	0·0000	−0·5000	−0·8000	−0·9000	−0·8000	−0·5000	0·0000	0·7000	1·6000
0·10	2·0000	1·1000	0·4000	−0·1000	−0·4000	−0·5000	−0·4000	−0·1000	0·4000	1·1000	2·0000
0·12	2·4000	1·5000	0·8000	0·3000	0·0000	−0·1000	0·0000	0·3000	0·8000	1·5000	2·4000
0·14	2·8000	1·9000	1·2000	0·7000	0·4000	0·3000	0·4000	0·7000	1·2000	1·9000	2·8000

20

t\x	0·00	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00
0·00	0·0000	30·9017	58·7785	80·9017	95·1057	100·0000	95·1057	80·9017	58·7785	30·9017	0·0000
0·04	0·0000	20·8224	39·6065	54·5136	64·0846	67·3825	64·0846	54·5136	39·6065	20·8224	0·0000
0·08	0·0000	14·0306	26·6878	36·7327	43·1818	45·4041	43·1818	36·7327	26·6878	14·0306	0·0000
0·12	0·0000	9·4542	17·9829	24·7514	29·0970	30·5944	29·0970	24·7514	17·9829	9·4542	0·0000
0·16	0·0000	6·3705	12·1174	16·6781	19·6063	20·6153	19·6063	16·6781	12·1174	6·3705	0·0000
0·20	0·0000	4·2926	8·1650	11·2381	13·2112	13·8911	13·2112	11·2381	8·1650	4·2926	0·0000
0·24	0·0000	2·8925	5·5018	7·5725	8·9021	9·3602	8·9021	7·5725	5·5018	2·8925	0·0000
0·28	0·0000	1·9490	3·7072	5·1026	5·9984	6·3071	5·9984	5·1026	3·7072	1·9490	0·0000
0·32	0·0000	1·3133	2·4980	3·4382	4·0419	4·2499	4·0419	3·4382	2·4980	1·3133	0·0000
0·36	0·0000	0·8849	1·6832	2·3168	2·7235	2·8637	2·7235	2·3168	1·6832	0·8849	0·0000
0·40	0·0000	0·5963	1·1342	1·5611	1·8352	1·9296	1·8352	1·5611	1·1342	0·5963	0·0000
0·44	0·0000	0·4018	0·7643	1·0519	1·2366	1·3002	1·2366	1·0519	0·7643	0·4018	0·0000
0·48	0·0000	0·2707	0·5150	0·7088	0·8332	0·8761	0·8332	0·7088	0·5150	0·2707	0·0000
0·52	0·0000	0·1824	0·3470	0·4776	0·5615	0·5904	0·5615	0·4776	0·3470	0·1824	0·0000
0·56	0·0000	0·1229	0·2338	0·3218	0·3783	0·3978	0·3783	0·3218	0·2338	0·1229	0·0000
0·60	0·0000	0·0828	0·1576	0·2169	0·2549	0·2680	0·2549	0·2169	0·1576	0·0828	0·0000

Test exercise 14 (page 614)

1 (a) $dz = 4x^3\cos 3y\,dx - 3x^4\sin 3y\,dy$ (b) $dz = 2e^{2y}\{2\cos 4x\,dx + \sin 4x\,dy\}$
(c) $dz = xw^2\{2yw\,dx + xw\,dy + 3xy\,dw\}$ **2** (a) $z = x^3y^4 + 4x^2 - 5y^3$
(b) $z = x^2\cos 4y + 2\cos 3x + 4y^2$ (c) not exact differential
3 9 square units **4** (a) 278·6 (b) $\pi/2$ (c) 22·5 (d) 48 (e) -21
(f) -54π **5** Area $= \frac{5}{12}$ square units **6** (a) 2 (b) 0

Further problems 14 (page 615)

1 14 **2** 1·6 **3** $\dfrac{\pi}{36}\{9 - 4\sqrt{3}\}$ **4** $\dfrac{1}{2}\{\pi^4 + 4\}$ **5** $\dfrac{9\pi}{256}$ **6** $\dfrac{1}{2}\cdot\ln 2$
7 $2 - \pi/2$ **8** $\dfrac{1}{8}$ **9** 14 **10** (a) 39·24 (b) 0 **11** $\dfrac{2}{3}$

Test exercise 15 (page 658)

1 $4\sqrt{2}\pi$ **2** $a(\pi/2)^2$ **3** (a) (1) (4·47, 0·464, 3) (2) (5·92, 0·564, 0·322)
(b) (1) (3·54, 3·54, 3) (2) $(-0{\cdot}832, 1{\cdot}82, 3{\cdot}46)$ **4** 12π
5 $a^3(8 - 3a)\pi/12$ **6** (a) $I = \displaystyle\iint v(1+u)(1+u+v)\,dv\,dv$
(b) $I = \displaystyle\iiint \frac{(2u+v)(v-4w)}{vw}\,du\,dv\,dw$

Further problems 15 (page 658)

1 $4\sqrt{5}\pi$ **2** $\left(\dfrac{a}{2}, \dfrac{a}{2}, \dfrac{a}{2}\right)$ **3** $10\sqrt{61}$ **4** $\dfrac{4\sqrt{22}\pi}{3}$ **5** $\dfrac{\pi}{24}(5\sqrt{5} - 1)$
6 $\pi\sqrt{5}$ **7** $16a^2$ **8** $2a^2(\pi - 2)$ **9** $4\pi(a+b)\sqrt{a^2 - b^2}$ **10** 45π

11 $\dfrac{11}{30}$ **12** $\dfrac{\pi a^4}{2}$ **13** $2\left(\pi - \dfrac{4}{3}\right)$ **14** $\bar{x} = \bar{y} = \bar{z} = \dfrac{3a}{8}$

15 $\dfrac{\pi a^3}{3}\{4\sqrt{2} - 3\}$ **16** $\dfrac{4\pi abc}{3}$ **17** $\dfrac{2a^3}{3}$ **18** $\dfrac{1}{4}\int\int (u^2 + v^2)\,du\,dv$

19 $u^2 v\,du\,dv\,dw$ **20** $\bar{z} = -\dfrac{a}{5}$ **21** $\dfrac{7}{18}$ **22** $2 - \dfrac{\pi}{2}$ **23** $\dfrac{1}{4}(\sqrt{2} - 1)$

Test exercise 16 (page 694)

1 (a) $\dfrac{20}{3}$ (b) $\dfrac{2}{3}$ (c) -2 (d) 120 (e) $\dfrac{15\sqrt{\pi}}{2048}$ **2** (a) $\dfrac{256}{315}$ (b) $\dfrac{1}{40}$ (c) $\dfrac{2}{105}$

3 (a) $\dfrac{1}{\sqrt{2}} \cdot K\left(\dfrac{1}{\sqrt{2}}\right)$ (b) $\dfrac{1}{2} \cdot F\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ **4** (a) 0 (b) 1 **5** (a) $F\left(\sqrt{2}, \dfrac{\pi}{4}\right)$

(b) $\dfrac{1}{2}F\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$

Further problems 16 (page 694)

1 (a) 6 (b) $-\dfrac{1}{2}$ (c) 0·4 (d) 24 (e) $\dfrac{315}{4}$ **2** (a) 6 (b) $\dfrac{8}{81}$ (c) $\dfrac{\sqrt{2}\pi}{16}$

(d) 4 **4** (a) $\dfrac{1}{8960}$ (b) $\dfrac{\sqrt{2}\pi}{64}$ (c) $\dfrac{8}{315}$ (d) $\dfrac{2}{7}$ (e) $\dfrac{1}{63}$ (f) $\dfrac{\pi}{432} = 0{\cdot}00727$

8 (a) $\sqrt{5} \cdot E\left(\dfrac{2}{\sqrt{5}}\right)$ (b) $\sqrt{2} \cdot K\left(\dfrac{1}{\sqrt{2}}\right) = 2{\cdot}622$ (c) $2 \cdot E\left(\dfrac{1}{2}, 1\right) = 2{\cdot}935$

(d) $\dfrac{1}{4} \cdot F\left(\dfrac{3}{4}, \dfrac{2}{3}\right) = 0{\cdot}193$ (e) $\dfrac{1}{\sqrt{5}} \cdot F\left(\dfrac{2}{\sqrt{5}}, 1\right)$ (f) $\dfrac{1}{\sqrt{2}} \cdot F\left(\dfrac{1}{\sqrt{2}}, \dfrac{\pi}{6}\right)$

(g) $\dfrac{1}{\sqrt{2}} \cdot \left\{F\left(\dfrac{1}{\sqrt{2}}, \dfrac{\pi}{3}\right) - F\left(\dfrac{1}{\sqrt{2}}, \dfrac{\pi}{4}\right)\right\}$ **9** $\dfrac{1}{2} \cdot \left\{F\left(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{2}\right) - F\left(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{4}\right)\right\}$

10 (a) $\dfrac{1}{\sqrt{3}} \cdot F\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{2}\right) = 0{\cdot}307$ (b) $\dfrac{1}{\sqrt{3}} \cdot \left\{F\left(\dfrac{1}{\sqrt{3}}, 1\right) - F\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{2}\right)\right\}$

(c) $\dfrac{1}{\sqrt{34}} \cdot K\left(\dfrac{3}{\sqrt{34}}\right) = 0{\cdot}2905$ (d) $\dfrac{1}{\sqrt{7}}\left\{F\left(\sqrt{\dfrac{3}{7}}, \dfrac{\pi}{2}\right) - F\left(\sqrt{\dfrac{3}{7}}, \dfrac{\pi}{6}\right)\right\}$

Text exercise 17 (page 741)

1 (a) -15 (b) $-16\mathbf{i} + 10\mathbf{j} + 17\mathbf{k}$ **2** (a) 9 (b) $-(47\mathbf{i} + 17\mathbf{j} + 29\mathbf{k})$

3 $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$ $\therefore$ vectors coplanar **4** (a) $4\mathbf{i} - 4\mathbf{j} + 24\mathbf{k}$

(b) $2\mathbf{i} - 2\mathbf{j} + 24\mathbf{k}$ (c) 24·66 **5** $\mathbf{T} = \dfrac{1}{\sqrt{66}}(4\mathbf{i} + \mathbf{j} + 7\mathbf{k})$

6 $\dfrac{8}{5}(25\mathbf{i} - 6\mathbf{j} - 15\mathbf{k})$ **7** 5·08 **8** $\dfrac{1}{\sqrt{101}}(2\mathbf{i} + 4\mathbf{j} + 9\mathbf{k})$

9 (a) $14\mathbf{i} - 12\mathbf{j} - 30\mathbf{k}$ (b) 8 (c) $5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ (d) $7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
(e) $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Further problems 17 (page 741)

1 61 **2** $29\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$ **3** (a) $22\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}$ (b) $-2\mathbf{i} + 14\mathbf{j} - 22\mathbf{k}$

4 (a) $2x\mathbf{i} + 3\mathbf{j} + \cos x\,\mathbf{k}$ (b) $2\mathbf{i} - \sin x\,\mathbf{k}$ (c) $(4x^2 + 9 + \cos^2 x)^{1/2}$
(d) $34 + \sin 2$ **5** (a) $2 - 2u - 9u^2$
(b) $(3u^2 + 4u + 3)\mathbf{i} + (3u^2 + 6)\mathbf{j} + (1 - 2u)\mathbf{k}$ (c) $\mathbf{i} - 2\mathbf{j} + (3 - 2u)\mathbf{k}$

6 $\dfrac{1}{5\sqrt{21}}(2\mathbf{i} - 20\mathbf{j} + 11\mathbf{k})$ **7** $\dfrac{-1}{\sqrt{129}}(10\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$

8 $\dfrac{-1}{\sqrt{126}}(5\mathbf{i}-\mathbf{j}+10\mathbf{k})$ **9** $\dfrac{-1}{\sqrt{601}}(12\mathbf{i}+4\mathbf{j}-21\mathbf{k})$ **10** $-8{\cdot}285$
11 $-9{\cdot}165$ **12** (a) 15 (b) -33 (c) 7 **13** (a) $-6\mathbf{i}+4\mathbf{j}-7\mathbf{k}$
(b) $62\mathbf{i}+10\mathbf{j}-38\mathbf{k}$ (c) $18\mathbf{i}-21\mathbf{j}+10\mathbf{k}$ **14** (a) $12\mathbf{i}-4\mathbf{j}+4\mathbf{k}$
(b) $24\mathbf{i}-4\mathbf{j}$ (c) 144 **15** (a) $(2\sin 2)\mathbf{i}+2e^3\mathbf{j}+(\cos 2+e^3)\mathbf{k}$
(b) $(4\sin^2 2+\cos^2 2+2e^3\cos 2+5e^6)^{1/2}$ **16** $-5{\cdot}014$
17 $p=\dfrac{1}{\sqrt{29}}(3\mathbf{i}+2\mathbf{j}-4\mathbf{k});\ q=\dfrac{1}{\sqrt{38}}(6\mathbf{i}-\mathbf{j}+\mathbf{k});\ \theta=68^\circ\,48'$
18 (a) $(2t+3)\mathbf{i}-(6\cos 3t)\mathbf{j}+6e^{2t}\mathbf{k}$ (b) $2\mathbf{i}+(18\sin 3t)\mathbf{j}+12e^{2t}\mathbf{k}$ (c) $12{\cdot}17$
20 $-4x\mathbf{i}+4z\mathbf{k}$ **21** $(2\cos 5{\cdot}5)\mathbf{i}-(6\sin 5{\cdot}5)\mathbf{j}-(6\sin 5{\cdot}5)\mathbf{k}$ **22** $p=6$
23 (a) (1) $p=15/4$ (2) $p=-33$ (b) $\dfrac{1}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$

Test exercise 18 (page 791)

1 $3\mathbf{i}+\dfrac{18}{7}\mathbf{j}-\dfrac{81}{8}\mathbf{k}$ **2** 12 **3** $18\pi(2\mathbf{i}+\mathbf{j})$ **4** $24(\mathbf{i}+\mathbf{j})$ **5** $8+\dfrac{4\pi}{3}$
6 all conservative **7** $36\left(\dfrac{\pi}{4}+1\right)$ **8** 0

Further problems 18 (page 792)

1 (a) $576\mathbf{k}$ (b) $\dfrac{576}{5}(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$ **2** $1771\mathbf{i}+1107\mathbf{j}+830{\cdot}4\mathbf{k}$
3 $416{\cdot}1\mathbf{i}+718{\cdot}5\mathbf{j}+5679\mathbf{k}$ **4** $46{\cdot}9$ **5** $-4{\cdot}18$ **6** 8π **7** $\dfrac{16\pi}{3}(\mathbf{i}+\mathbf{k})$
8 $\dfrac{1}{3}(48\mathbf{i}+64\mathbf{j}-24\mathbf{k})$ **9** $64\left(\dfrac{\pi}{4}-\dfrac{1}{3}\right)(6\mathbf{i}+4\mathbf{j})$
10 $\dfrac{9}{2}\{(\pi+2)\mathbf{i}+(\pi+2)\mathbf{j}+4\mathbf{k}\}$ **11** $\dfrac{12}{5}(32\mathbf{j}+15\mathbf{k})$ **12** -1 **13** $\dfrac{250}{3}\pi$
14 $\dfrac{1}{6}(117\pi+256-28\sqrt{7})=91{\cdot}58$ **15** -80 **16** 96π **17** -2 **18** 12π
19 $-\dfrac{a^3}{3}$ **20** $\dfrac{81\pi}{4}$

Test exercise 19 (page 819)

1 yes, an orthogonal set **2** $h_u=1,\ h_v=2v,\ h_\theta=2u$ **3** $4\mathbf{I}+\mathbf{K}$
4 (a) $(2\cos\phi+2\cos 2\phi+1)$ (b) $(2\sin 2\phi+\sin\phi)\mathbf{K}$
5 (a) $(\mathrm{d}s)^2=(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2+r^2\sin^2\theta\,(\mathrm{d}\phi)^2$ (b) $\mathrm{d}V=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
6 $-10{\cdot}5$

Further problems 19 (page 820)

1 (a) yes (b) no **2** $-50{\cdot}5$ **3** $2\dfrac{5}{18}$
5 (a) $\nabla^2 V=\dfrac{\partial^2 V}{\partial\rho^2}+\dfrac{1}{\rho}\cdot\dfrac{\partial V}{\partial\rho}+\dfrac{1}{\rho^2}\cdot\dfrac{\partial^2 V}{\partial\phi^2}+\dfrac{\partial^2 V}{\partial z^2}$
(b) $\nabla^2 V=\dfrac{1}{r^2}\cdot\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\cdot\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\cdot\dfrac{\partial^2 V}{\partial\phi^2}$
6 (b) $h_u=h_v=\sqrt{u^2+v^2};\ h_w=1$

(c) $\operatorname{div} F = \dfrac{1}{u^2+v^2}\left\{\dfrac{\partial}{\partial u}\left(\sqrt{u^2+v^2}\cdot\dfrac{\partial F_u}{\partial u}\right)+\dfrac{\partial}{\partial v}\left(\sqrt{u^2+v^2}\cdot\dfrac{\partial F_v}{\partial v}\right)\right\}+\dfrac{\partial F_w}{\partial w}$

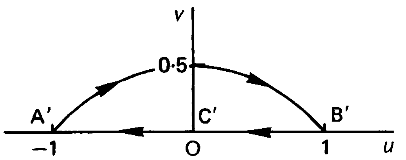
(d) $\nabla^2 V = \dfrac{1}{u^2+v^2}\left\{\dfrac{\partial^2 V}{\partial u^2}+\dfrac{\partial^2 V}{\partial v^2}\right\}+\dfrac{\partial^2 V}{\partial w^2}$

Test exercise 20 (page 858)

1 (a) $w = 6 - j2$ (b) $w = 3 - j2$ (c) $w = j3$ (d) $w = 2$

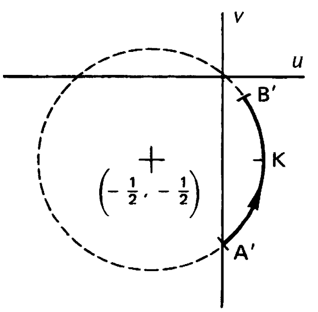
2 Magnification = 2·236; rotation = 63° 26′; translation = 1 unit to right, 3 units downwards

3



$v = \dfrac{1}{2}(1-u^2)$

4



Minor arc of circle, centre $\left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$, radius $\dfrac{1}{\sqrt{2}} = 0{\cdot}7071$, between A′ $(-j)$ and B $(0{\cdot}12 - j0{\cdot}16)$

5 (a) centre $\left(u = 0, v = \dfrac{2}{3}\right)$ (b) radius $\dfrac{1}{3}$

6 centre $\left(u = \dfrac{2}{3}, v = 0\right)$; radius $\dfrac{2}{3}$

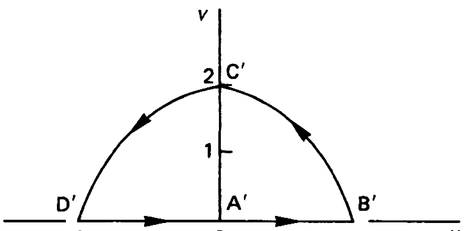
Further problems 20 (page 859)

1 Triangle A′B′C′ with A′ $(-1+j2)$, B′ $(5+j2)$, C′ $(2+j5)$

2 (a) A′ $(-8+j9)$; B′ $(23+j14)$
(b) Magnification = $\sqrt{29}$ = 5·385; rotation = 68°12′; translation = nil

3 Straight line joining A′ $(5-j7)$ to B′ $(-3-j)$; magnification = 3·162; rotation = 161° 34′ anticlockwise; translation = 2 to right, 4 upwards

4



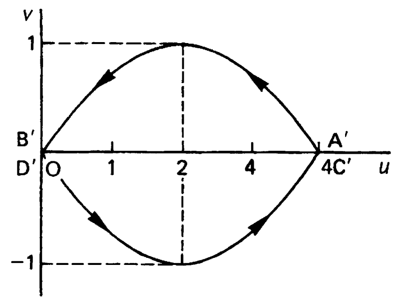
A′ B′: $v = 0$

B′C′: $u = 1 - \dfrac{v^2}{4}$

C′D′: $u = \dfrac{v^2}{4} - 1$

D′A′: $v = 0$

5



A′B′ and C′D′: $v = \frac{1}{4}(4u - u^2)$

B′C′ and D′A′: $v = \frac{1}{4}(u^2 - 4u)$

6 A′ $(1 - j2)$; B′ $(-23 + j10)$; C′ $(1 - j8)$ A′B′: $u = 2 - \frac{v^2}{4}$;
B′C′: $v = \frac{(u-1)^2}{32} - 8$; C′A′: $u = 1$ **7** circle, centre $\left(\frac{1}{2} - j\frac{2}{3}\right)$, radius $\frac{7}{6}$

8 (a) circle, centre $\left(\frac{1}{3} - j0\right)$, radius $\frac{2}{3}$ (b) region outside the circle in (a)

9 circle, centre $\left(\frac{3}{2} + j0\right)$, radius 1; clockwise development

10 circle, $u^2 + v^2 - \frac{22u}{5} + \frac{8}{5} = 0$, centre $\left(\frac{11}{5} + j0\right)$, radius $\frac{9}{5}$

11 circle, $u^2 + v^2 - \frac{u}{2} = 0$, centre $\left(\frac{1}{4} + j0\right)$, radius $\frac{1}{4}$; region inside this circle

12 circle, centre $\left(-\frac{7}{3} + j0\right)$, radius $\frac{5}{3}$

13 (a) circle, centre $\left(\frac{3}{5}, 0\right)$, radius $\frac{2}{5}$, developed clockwise
(b) region outside the circle in (a)

14 $v = -\frac{u}{3}$

Test exercise 21 (page 906)

1 (a) regular at all points (b) $z = -5$ (c) regular at all points
(d) $z = -1$ and $z = 4$ (e) $z = 0$, where $z = x + jy$
2 (a) $v(x, y) = \cosh x \cos y + C$ (b) $v(x, y) = 6(y^2 - x^2) - 4x + C$ **4** $j4\pi$
5 (a) $z = 0$ (b) $z = \pm 1$ (c) no critical point (d) $z = \pm\sqrt{2}$ (e) $z = 0$
(f) no critical point **6** $w = \cosh\frac{\pi z}{4}$; D′: $w = 1$

Further problems 21 (page 907)

3 circle, centre $(5, -2)$, radius $\sqrt{2}$ **4** circle, centre $\left(-\frac{1}{3}, 0\right)$,
radius $\frac{2}{3}$, anticlockwise **5** (a) $v(x, y) = 2y(x - 1) + C$
(b) $v(x, y) = 3x^2y - y^3 - 2xy + y + C$ (c) $v(x, y) = x^2 - 2x - y^2 + C$
(d) $v(x, y) = e^{x^2 - y^2}\sin 2xy + C$ **6** (a) $j10\pi$ (b) $j6\pi$ **7** (a) 0 (b) $j4\pi$
(c) $j10\pi$ **9** $j2\pi$ **10** $j10\pi$ **11** (a) (1) $z = 0$ (2) $z = \pm 1$
(b) ellipse, centre (0, 0), semi major axis $\frac{5}{2}$, semi minor axis $\frac{3}{2}$

12 (a) $u^2+v^2=1$ (b) $u^2+(v-1)^2=2$; $\theta=45°$. **13** Unit circle becomes the real axis on the w-plane. Region within the circle maps onto the upper half plane **14** $w=\sin\dfrac{z\pi}{2a}$

Test exercise 22 (page 936)

1 (a) $f(z)=1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\ldots+\dfrac{z^n}{n!}+\ldots$ valid for $|z|<\infty$

(b) $f(z)=4z-\dfrac{(4z)^2}{2}+\dfrac{(4z)^3}{3}-\ldots+\dfrac{(-1)^{n+1}(4z)^n}{n}+\ldots$ valid for $|z|<1/4$ **2** (a) pole of order 5 at $z=-1$ (b) essential singularity at $z=0$ (c) essential singularity at $z=0$ (d) removable singularity at $z=0$

3 $$f(z)=\frac{1}{\sqrt{2}}\left\{1+(z-\pi/4)-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+\frac{(z-\pi/4)^4}{4!}+\frac{(z-\pi/4)^5}{5!}-\ldots\right\};$$ valid for $|z|<\infty$

4 (a) $f(z)=-(z+3)+8+\dfrac{1}{2(z+3)}-\dfrac{4}{(z+3)^2}-\dfrac{1}{24(z+3)^3}+\dfrac{1}{3(z+3)^4}+\ldots;$ essential singularity

(b) $f(z)=\dfrac{3}{z+3}-\dfrac{1}{z+1}=\ldots-\dfrac{1}{z^3}+\dfrac{1}{z^2}-\dfrac{1}{z}+1-\dfrac{z}{3}+\dfrac{z^2}{9}-\dfrac{z^3}{27}+\ldots$

(c) $f(z)=\dfrac{1}{8(z-2)^2}-\dfrac{3}{16(z-2)}+\dfrac{3}{16}-\dfrac{5(z-2)}{32}+\dfrac{15(z-2)^2}{64}+\ldots$; pole of order 2 **5** double pole at $z=0$; residue -4, double pole at $z=-1$, residue $7/2$, single pole at $z=1$, residue $1/2$ **6** (a) $-\pi/6$ (b) $2\pi/\sqrt{3}$ (c) $2\pi e^{-3}$

Further problems 22 (page 937)

1 (a) $z+\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+\ldots+\dfrac{z^{2n+1}}{(2n+1)!}+\ldots$, $|z|<\infty$

(b) $z+\dfrac{z^3}{3}+\dfrac{2z^5}{15}+\dfrac{17z^7}{315}+\ldots$, $|z|<\pi/2$

(c) $2\left\{z+\dfrac{z^3}{3}+\dfrac{z^5}{5}+\ldots+\dfrac{z^{2n+1}}{2n+1}+\ldots\right\}$, $|z|<1$

(d) $1+z\ln a+\dfrac{z^2(\ln a)^2}{2!}+\dfrac{z^3(\ln a)^3}{3!}+\ldots+\dfrac{z^n(\ln a)^n}{n!}+\ldots$, $|z|<\infty$

(e) $\dfrac{3z^2}{25}+\dfrac{27z^3}{125}+\dfrac{162z^4}{625}+\dfrac{810z^5}{3125}+\ldots$, $|z|<5/3$;

$-\dfrac{5}{9z}-\dfrac{25}{9z^2}-\dfrac{250}{27z^3}-\dfrac{6250}{243z^4}-\ldots$, $|z|>5/3$ **3** (b) $-\dfrac{z}{(z+1)^2}, \dfrac{z(z-1)}{(z+1)^3}$

4 (a) convergent for $|z|<\infty$ (b) convergent for $|z|<1$ (c) convergent for $|z|<1$ (d) convergent for $|z|<1$ (e) convergent for $|z|<\infty$

5 (a) $e^2\left\{1+(z-2)+\dfrac{(z-2)^2}{2!}+\dfrac{(z-2)^3}{3!}+\ldots+\dfrac{(z-2)^n}{n!}+\ldots\right\}$

(b) $\dfrac{\sqrt{3}}{2}-\dfrac{(z-\pi/6)}{2}-\dfrac{\sqrt{3}(z-\pi/6)^2}{2\times 2!}+\dfrac{(z-\pi/6)^3}{2\times 3!}+\dfrac{\sqrt{3}(z-\pi/6)^4}{2\times 4!}+\ldots$

(c) $(z-3)\sin 6 + (z-3)^2\cos 6 - \dfrac{(z-3)^3\sin 6}{2!} - \dfrac{(z-3)^4\cos 6}{3!}$
$+\dfrac{(z-3)^5\sin 6}{4!} + \dots$ (d) $-\left\{\dfrac{3}{13} + 2\left(\dfrac{3}{13}\right)^2(z-1/3)\right.$
$\left.+4\left(\dfrac{3}{13}\right)^3(z-1/3)^2 + \dots + 2^n\left(\dfrac{3}{13}\right)^{n+1}(z-1/3)^n + \dots\right\}$
(e) $1 - 2(z-3) + 4(z-3)^2 + \dots + (-2)^n(z-3)^n + \dots$

6 $(z-1) + \dfrac{(z-1)^2}{1\times 2} - \dfrac{(z-1)^3}{2\times 3} + \dfrac{(z-1)^4}{3\times 4} - \dfrac{(z-1)^5}{4\times 5} + \dots$ **7** (a) $z = \infty$ (b) $|z| = \sqrt{6}$ (c) $|z-5| = 1$ (d) $z = \infty$ **8** (a) poles of order 2 at $z = 0$ and $z = -1$, removable singularity at $z = \pm 1$ (b) essential singularity at $z = 0$ **9** (a) $\dfrac{1}{z^2} - \dfrac{1}{z^4 3!} + \dfrac{1}{z^6 5!} - \dfrac{1}{z^8 7!} + \dots$, $|z| > 0$
(b) $\dfrac{1}{2}\left(z - \dfrac{3}{2}\right)^{-1}$, $|2z-3| > 0$
(c) $\dfrac{3}{z-3} - 2\{1 - (z-3) + (z-3)^2 - (z-3)^3 + \dots\}$, $0 < |z-3| < 1$

10 (a) $\dots + \dfrac{8}{z^4} - \dfrac{4}{z^3} + \dfrac{2}{z^2} - \dfrac{1}{z} + \dfrac{2}{5} - \dfrac{2z}{25} + \dfrac{2z^2}{125} - \dfrac{2z^3}{625} + \dots$
(b) $\dfrac{1}{z} - \dfrac{8}{z^2} + \dfrac{46}{z^3} - \dfrac{242}{z^4} + \dots$ (c) $-\dfrac{1}{10} + \dfrac{17z}{100} - \dfrac{109z^2}{1000} + \dfrac{593z^3}{10000} - \dots$

11 (a) $2\pi/\sqrt{3}$ (b) $\dfrac{2\pi}{\sqrt{\alpha^2-\beta^2}}$ (c) $\dfrac{2\pi}{|\alpha^2-1|}$ (d) $\pi/4$ (e) $\pi/2$ (f) $\pi/2$
(g) $\pi\sqrt{\sqrt{13}/8 - 3/8}$ (h) $\pi/4$ (i) $2\pi/\sqrt{3}$ (j) $2\pi/3$ (k) 0 (l) 0

Test exercise 23 (page 983)

1 $P_{max} = 10$ $(x = 4,\ y = 3)$ **2** $P_{max} = 13$ $(x = 1,\ y = 4)$
3 $P_{max} = 188$ $(x = 10, y = 4, z = 6)$ **4** $P_{max} = 296$ $(x = 4,\ y = 6)$
5 $P_{min} = 16$ $(x = 5,\ y = 12)$ **6** (a) 13 type A + 4 type B (b) £11,800

Further problems 23 (page 984)

1 $P_{max} = 32$ $(x = 4,\ y = 9/2)$ **2** $P_{max} = 64$ $(x = 0,\ y = 8)$
3 $P_{max} = 40$ $(x = 6,\ y = 5/2)$ **4** $P_{max} = 15$ $(x = 6,\ y = 3)$
5 $P_{max} = 9$ $(x = 1,\ y = 3)$ **6** $P_{max} = 10$ $(x = 2,\ y = 4)$
7 $P_{max} = 10$ $(x = 2,\ y = 4)$ **8** $P_{max} = 37$ $(x = 0,\ y = 8,\ z = 1)$
9 $P_{max} = 67$ $(x = 4,\ y = 10,\ z = 5)$ **10** $P_{max} = 65$ $(x = 5,\ y = 10,\ z = 5)$
11 $P_{max} = 11{\cdot}568$ $(x = 29/22,\ y = 14/11,\ z = 0)$ to 3 *s.f.*
12 $P_{max} = 340$ $(x = 30,\ y = 20)$ **13** $P_{max} = 112$ $(x = 4,\ y = 8)$
14 $P_{max} = 108$ $(x = 16,\ y = 15)$ **15** $P_{min} = 138$ $(x = 12,\ y = 18)$
16 $P_{max} = 240$ $(x = 9,\ y = 15)$ **17** $P_{max} = 4400$ $(x = 201,\ y = 53)$
18 $P_{max} = 100$ $(x = 20,\ y = 10)$ **19** $P_{max} = 410$ $(x = 9,\ y = 5,\ z = 2)$
20 $P_{max} = 1560$ $(x = 11,\ y = 10,\ z = 18)$
21 $P_{max} = 660$ $(x = 60,\ y = 30,\ z = 30)$ **22** $P_{min} = -14$ $(x = 5,\ y = 2)$
23 $P_{min} = 56$ $(x = 8,\ y = 12)$ **24** $P_{min} = 16$ $(x = 8,\ y = 6)$
25 $P_{min} = 40$ $(x = 4,\ y = 4)$ **26** $P_{min} = -10$ $(x = 6,\ y = 13,\ z = 14)$
27 $P_{min} = -75$ $(x = 8,\ y = 12,\ z = 21)$
28 (a) 10 type A + 35 type B (b) £2150
29 (a) 22 type A + 44 type B + 48 type C (b) £12,580
30 (a) 129 type A + 0 type B + 185 type C; (b) £8955

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